Robust Estimation for Discrete-Time Markovian Jump Linear Systems

http://www.producao.usp.br/handle/BDPI/46590

Downloaded from: Biblioteca Digital da Produção Intelectual - BDPI, Universidade de São Paulo
Robust Estimation for Discrete-Time Markovian Jump Linear Systems

Marco H. Terra, João Y. Ishihara, Gildson Jesus, and João P. Cerri

Abstract—This technical note deals with recursive robust estimation of Markovian jump linear systems subject to unobserved chain state. It is developed an augmented system modeled via norm bounded uncertainties. Upper and lower bounds of an uncertain quadratic cost function are computed in order to define the variances of the augmented system. Numerical examples are provided to show the effectiveness of the proposed estimators.

Index Terms—Discrete-time, Markovian systems, robust estimation.

I. INTRODUCTION

State estimation of discrete-time Markovian jump linear systems (DMJLS) has been considered in some important applications related with robotics, communication, finance (see, e.g., [1]–[7] and references therein). Nominal and robust estimators have been investigated by several authors in the literature (see, e.g., [8]–[15]). Optimal and suboptimal schemes based on maximum likelihood estimates have been proposed as extensions of the original Kalman filter proposed in the 1960s.

An interesting solution for state estimation of DMJLS is the recursive approach proposed in [11], since it is well suited for online implementations. However, it does not present robustness properties. The DMJLS parameters are not subject to uncertainties. To overcome this problem, some approaches have been proposed in the literature. A well established technique is based on H∞ criteria, which consider robustness against disturbances and parametric uncertainties. The solutions provided by this class of robust estimators are based on linear matrix inequalities (LMIs) (see, e.g., [16]–[24]). Due to possible infeasible solutions inherent to LMIs, all computations should be performed offline before these estimators be implemented. On the other hand, standard robust Kalman-type estimators developed for systems not subject to Markovian jumps (see, e.g., [25] and [26]) cannot be directly applied to estimate uncertain DMJLS when the Markov chain is not known.

In this technical note, recursive robust state estimators (in the predicted and filtered forms) for DMJLS are developed based on discrete-time Riccati recursions. They are deduced through the minimization of the worst-possible regularized residual norm. The main difficulty to solve this kind of problem is to find an appropriate cost function considering the Markov chain not available. An advantage of the proposed robust filters for DMJLS is that the conditions for stability can be easily stated and proved.

An important contribution of this note resides in the variance computation of the disturbance related with the states of an augmented DMJLS. The computation of this variance is possible thanks to the formulas proposed to calculate upper and lower bounds of an uncertain quadratic term. They are obtained through the solution of optimization problems.

Simulations are performed for the robust predictor developed, the stationary predictor proposed in [11] and the robust predictor subject to polytopic uncertainties given in [16], for comparison purposes. It is emphasized in this comparative study the importance of the recursive-ness for online applications. For the case with no jumps, the predicted and the filtered robust states developed reduce to the recursive robust estimators given in [25]. In addition, if the system is considered without uncertainties, both estimates reduce to the respective standard Kalman-type estimators in the predicted and filtered forms.

This technical note is organized as follows: in Section II, the uncertain DMJLS and its augmented version are presented; in Section III, robust filters for DMJLS are deduced; and in Section IV, numerical examples are shown.

II. PRELIMINARIES

Consider the following uncertain DMJLS:

\[
\begin{align*}
x_{i+1} &= (F_i + \delta F_i) x_i + G_i \omega_i, & i = 0, 1, \ldots \\
y_i &= (H_i + \delta H_i) x_i + J_i \nu_i 
\end{align*}
\]

where, for each time instant \(i\) and jump parameter \(\Theta_i \in S, S := \{1, \ldots, N\}\), \(F_{i, \Theta_i}, G_{i, \Theta_i}, H_{i, \Theta_i}, J_{i, \Theta_i}\), are nominal parameter matrices subject to the uncertainties

\[
\begin{align*}
\Delta F_{i, \Theta_i} &= M^F_{i, \Theta_i} \Delta^F_{\Theta_i} N^F_{i, \Theta_i}, \quad ||\Delta^F_{\Theta_i}|| \leq 1 \\
\Delta H_{i, \Theta_i} &= M^H_{i, \Theta_i} \Delta^H_{\Theta_i} N^H_{i, \Theta_i}, \quad ||\Delta^H_{\Theta_i}|| \leq 1
\end{align*}
\]

with \(\Delta^F_{\Theta_i}\) and \(\Delta^H_{\Theta_i}\) arbitrary matrices; \(M^F_{i, \Theta_i}, N^F_{i, \Theta_i}, M^H_{i, \Theta_i}, N^H_{i, \Theta_i}\) known matrices of appropriate dimensions. Assume that \(\Theta := \{\Theta_i, i = 0, 1, \ldots \mid \Theta_i \in S\}\) is a finite state discrete-time Markov chain with \(\pi_{i,j} := P(\Theta_i = j)\) and state transition probability matrix \(P = [p_{jk}] \in \mathbb{R}^{N \times N}\) whose entries are given by \(p_{jk} := P(\Theta_{i+1} = k \mid \Theta_i = j)\) for all \(k, j \in S\).

In this technical note, recursive robust state estimators (in the predicted and filtered forms) for DMJLS are developed based on discrete-time Riccati recursions. They are deduced through the minimization of the worst-possible regularized residual norm. The main difficulty to solve this kind of problem is to find an appropriate cost function considering the Markov chain not available. An advantage of the proposed robust filters for DMJLS is that the conditions for stability can be easily stated and proved.

An important contribution of this note resides in the variance computation of the disturbance related with the states of an augmented DMJLS. The computation of this variance is possible thanks to the formulas proposed to calculate upper and lower bounds of an uncertain quadratic term. They are obtained through the solution of optimization problems.

Simulations are performed for the robust predictor developed, the stationary predictor proposed in [11] and the robust predictor subject to polytopic uncertainties given in [16], for comparison purposes. It is emphasized in this comparative study the importance of the recursive-ness for online applications. For the case with no jumps, the predicted and the filtered robust states developed reduce to the recursive robust estimators given in [25]. In addition, if the system is considered without uncertainties, both estimates reduce to the respective standard Kalman-type estimators in the predicted and filtered forms.

This technical note is organized as follows: in Section II, the uncertain DMJLS and its augmented version are presented; in Section III, robust filters for DMJLS are deduced; and in Section IV, numerical examples are shown.
From the augmented system (3), one can show that
\[ \mathcal{M}_{i+1} = \mathcal{M}_{i+1}^a, \quad \mathcal{P}_{i+1} = \mathcal{P}_{i+1}^a, \]
for each \( i \), the second-order moments of the augmented system are described by the following variables defined for \( i \) and \( k \), where \( \mathcal{M}_{i+1} \) is updated through the equation (4) with initial condition (5) and (6).

The state of the original system (1) is recovered as
\[ x_i = \sum_{j=1}^{N} z_{i,j}. \]

According to [2], when (3) is not subject to uncertainties, the second-order moments \( \{Z_i^N\} \) and \( \{P_i^N\} \) are calculated through the following equations:
\[ Z_{i+1}^N = \sum_{j=1}^{N} p_{j,k} F_j Z_{i,j}^N + \sum_{j=1}^{N} p_{j,k} \pi_{i,j} G_{i,j} U_i G_{i,j}^T \]
\[ P_{i+1}^N = \text{diag} \left( \sum_{j=1}^{N} p_{j,k} F_j Z_{i,j}^N F_j^T \right) - \mathcal{F} Z_{i+1}^N \mathcal{F}^T + \text{diag} \left( \sum_{j=1}^{N} p_{j,k} \pi_{i,j} G_{i,j} U_i G_{i,j}^T \right). \]

The covariance \( \mathcal{R}_i \) is easily computed through (6). However, it is difficult to use (4) and (5) in their present forms, they depend on \( \mathcal{F}_i \) and \( \delta \mathcal{F}_i \). In the next section, this problem is dealt with.

### III. ROBUST FILTERS FOR DMJLS

In order to develop robust filters for the DMJLS (3), it is needed to define an upper bound to the covariance \( P_i \) given in (5). It is composed by the difference of positive and negative uncertain terms. In [27] is provided a formula, which was deduced from inequality arguments, to compute upper bounds of uncertain quadratic terms. On the other hand, it was not found in the literature an expression to calculate lower bounds, necessary to compute the lower influence of \( \mathcal{L}_i \) in (5). In the next proposition and corollary, lower and upper bounds based on optimizations of a suitable cost function are proposed.

Proposition 1: Let the following quadratic cost function:
\[ J(x,y) = (A x - b + H y)^T W (A x - b + H y), \]

where \( A \in \mathbb{R}^{n \times m}, b \in \mathbb{R}^n, H \in \mathbb{R}^{n \times p} (H \neq 0) \) and \( W \in \mathbb{R}^{n \times n} (W \geq 0) \) are assumed known, and \( x \in \mathbb{R}^n, y \in \mathbb{R}^p \) are unknown variables. For any fixed \( x \), consider the following optimization problems subject to \( y \) given by:
\[ \min_{y \leq \phi(x)} \{ J(x,y) \} \quad \text{and} \quad \max_{y \leq \phi(x)} \{ J(x,y) \}, \]

where \( \phi : \mathbb{R}^n \rightarrow \mathbb{R}^+ \) is a nonnegative function. Define \( I_i(x) := \min_{\|y\| \leq \phi(x)} \{ J_i(x,y) \} \) and \( \mathcal{L}^*(x) := \max_{\|y\| \leq \phi(x)} \{ J_i(x,y) \} \). Then,
\[ I_i(x) \leq J_i(x) \leq \mathcal{L}^*(x) \]
for all \( y \) such that \( \|y\| \leq \phi(x) \), where
\[ I_i(x) := (A x - b)^T W (A x - b + H y_i(x)) \quad \text{and} \quad \mathcal{L}^*(x) := (A x - b)^T W (A x - b + H y^*_i(x)) \]
with
\[ W_i(x) := \left( W - W H (A I + H^T W)^{-1} H^T W \right) \]
\[ W_n(x) := \left( W + W H (A I - H^T W)^{-1} H^T W \right) \]
and \( \lambda > \|H^T W\| \).
Proof: It follows the arguments used in [28], which are based on Lagrange multipliers.

The next corollary defines upper and lower bounds in a way useful to be applied to the estimators considered in this note.

Corollary 1: Consider the following quadratic term:

$$J(\delta F) = (F + \delta F)^T X(F + \delta F)$$

with $$\delta F = -H \Delta E \frac{\Delta}{|\Delta|} \leq 1$$, where $$F \in \mathbb{R}^{n \times m}$$, $$X \in \mathbb{R}^{n \times m}$$, $$\Delta \in \mathbb{R}^{n \times p}$$, and $$H \in \mathbb{R}^{n \times p} (H \neq 0)$$, and $$F_P \in \mathbb{R}^{p \times n} (F_P \neq U)$$ are assumed known and $$\Delta \in \mathbb{R}^{n \times p}$$. Then, for all $$\delta F$$, the following inequalities hold:

$$F X_i(\lambda) E^T - \lambda H H^T \leq J(\delta F) \leq F X_u(\lambda) E^T + \lambda H H^T$$

with

$$X_i(\lambda) := \left( X - X E^T (\lambda I + E_P X E_P^T)^{-1} E_P X \right)$$

$$X_u(\lambda) := \left( X + X E^T (\lambda I + E_P X E_P^T)^{-1} E_P X \right)$$

and $$\lambda > \|F P X E_P^T\|$$. 

Proof: It follows from Proposition 1, through some appropriate identifications. For all $$x$$ arbitrarily chosen, one has

$$J(\delta F) = x^T (F + \delta F) X(F + \delta F) x$$

$$= \left( F x + E_P^T \Delta H^T x \right)^T (F x + E_P^T \Delta H^T x)$$

Consider then the following identifications: $$A = F^T$$, $$b = 0$$, $$H = E_P^T$$, $$\gamma = \Delta H^T x$$, $$W = X$$, based on the quadratic cost function (7).

Recalling that $$|\Delta| \leq 1$$, it is easy to check that $$\|g\| = \|F^T \Delta H^T x\| \leq \|\Delta\| \|\Delta H^T x\| \leq |\Delta H^T x|$$, with $$\phi(x) := \|H^T x\|$$. Considering Proposition 1, one obtains the result. □

Based on the Corollary 1, one can calculate bounds to (4) and (5). For the uncertain covariance sequence $$(Z_i) = \{diag[Z_i]\}$$ generated by (4), the lower and upper bounds $$(Z_i^L) = \{diag[Z_i^L]\}$$ and $$(Z_i^U) = \{diag[Z_i^U]\}$$, respectively, are generated by the following recursive equations:

$$Z_{i+1}^L = \sum_{j=1}^{N} F_j \left( \left( Z_i^L \right)^{-1} + \sigma_{i,j}^2 N_i^T N_j^T \right)^{-1} F_j^T$$

$$- \kappa_{i,j} M_{i,j} M_{j,i}^T + \sigma_{i,j} G_{i,j} U_i G_{j,i}^T$$

(8)

$$Z_{i+1}^U = \sum_{j=1}^{N} F_j \left( \left( Z_i^U \right)^{-1} - \epsilon_{i,j} N_i^T N_j^T \right)^{-1} F_j^T$$

$$+ \epsilon_{i,j} M_{i,j} M_{j,i}^T + \sigma_{i,j} G_{i,j} U_i G_{j,i}^T$$

(9)

with $$Z_{i+1}^L = Z_i^U = \Xi_k$$. The parameters $$\kappa_{i,j}$$ and $$\epsilon_{i,j} \in \mathbb{R}^+$$ are given by

$$\kappa_{i,j} = \beta_{i,j} \sigma_{i,j} \left( F_{j,k} \sigma_{i,j} G_{i,j} U_i G_{j,k}^T \right) \times \sigma_{\min}(P_{j,k}, M_{j,k}^T M_{j,k})^{-1}$$

where $$0 < \beta_{i,j} < 1$$ and $$\epsilon_{i,j} = \alpha_{i,j} \left\|N_i^T Z_i^L N_i^T \right\|$$ with $$\alpha_{i,j} > 1$$; $$\sigma_{\min}(X)$$ and $$\sigma_{\max}(X)$$ stand for the minimum and maximum singular values of $$X$$, respectively.

It can be shown that for each $$i$$ ($$i = 0, 1, \ldots$$), the covariances $$Z_i^L$$, $$Z_i^U$$, $$Z_i^N$$, and $$Z_i^U$$ are related as

$$Z_i^L \leq Z_i \leq Z_i^U$$

for all $$i$$ such that $$\Delta_i \leq 1$$; for all $$i$$ such that $$\kappa_{i,i} > 1$$; and for all $$i$$ such that $$\epsilon_{i,i} > 0$$ such that $$\epsilon_{i,i} I - N_i^T Z_i^L N_i^T > 0$$. In particular, if $$M_i = 0$$ and $$N_i = 0$$ then

$$Z_i^L = Z_i = Z_i^N = Z_i^U$$.

Now, one can calculate an upper bound for (5) based on (8) and (9), as shown in (10). One can compute recursively (8) whose stability property is similar to the standard Riccati equations [29]. However, the stability of (9) is not guaranteed for any $$\epsilon_{i,j} = -\alpha_{i,j} \left\|N_i^T Z_i^U N_i^T \right\|$$.

Based on the Corollary 1, one can calculate bounds to (4) and (5). For the uncertain covariance sequence $$(Z_i) = \{diag[Z_i]\}$$ generated by (4), the lower and upper bounds $$(Z_i^L) = \{diag[Z_i^L]\}$$ and $$(Z_i^U) = \{diag[Z_i^U]\}$$, respectively, are generated by the following recursive equations:

$$Z_{i+1}^L = \sum_{j=1}^{N} F_j \left( \left( Z_i^L \right)^{-1} + \sigma_{i,j}^2 N_i^T N_j^T \right)^{-1} F_j^T$$

$$- \kappa_{i,j} M_{i,j} M_{j,i}^T + \sigma_{i,j} G_{i,j} U_i G_{j,i}^T$$

(8)

$$Z_{i+1}^U = \sum_{j=1}^{N} F_j \left( \left( Z_i^U \right)^{-1} - \epsilon_{i,j} N_i^T N_j^T \right)^{-1} F_j^T$$

$$+ \epsilon_{i,j} M_{i,j} M_{j,i}^T + \sigma_{i,j} G_{i,j} U_i G_{j,i}^T$$

(9)

It can be shown that for each $$i$$ ($$i = 0, 1, \ldots$$), the covariances $$Z_i^L$$, $$Z_i^U$$ are related as

$$Z_i^L \leq Z_i \leq Z_i^U$$

for all $$i$$ such that $$\Delta_i \leq 1$$; for all $$i$$ such that $$\kappa_{i,i} > 1$$; and for all $$i$$ such that $$\epsilon_{i,i} > 0$$ such that $$\epsilon_{i,i} I - N_i^T Z_i^L N_i^T > 0$$. In particular, if $$M_i = 0$$ and $$N_i = 0$$ then

$$Z_i^L = Z_i = Z_i^N = Z_i^U$$.

For all $$i$$ such that $$\Delta_i \leq 1$$; for all $$i$$ such that $$\kappa_{i,i} > 1$$; and for all $$i$$ such that $$\epsilon_{i,i} > 0$$ such that $$\epsilon_{i,i} I - N_i^T Z_i^L N_i^T > 0$$. In particular, if $$M_i = 0$$ and $$N_i = 0$$ then

$$Z_i^L = Z_i = Z_i^N = Z_i^U$$.

Now, one can calculate an upper bound for (5) based on (8) and (9), as shown in (10). One can compute recursively (8) whose stability property is similar to the standard Riccati equations [29]. However, the stability of (9) is not guaranteed for any $$\epsilon_{i,j} = -\alpha_{i,j} \left\|N_i^T Z_i^U N_i^T \right\|$$.

The computation of upper bounds of uncertain systems has been the subject of several authors, see for instance [30]–[33]. The approaches considered in these references are based on only LMIs. In the following, the computation of $$Z_i$$, performed in two steps. In the first, $$Z_i^L$$ is obtained through a minimization problem and in the second, $$Z_i^U$$

$$P_i \leq \Pi_i := diag \left\{ \sum_{j=1}^{N} P_{i,j} \left( F_j \left( \left( Z_i^L \right)^{-1} + \sigma_{i,j}^2 N_i^T N_j^T \right)^{-1} F_j^T + \kappa_{i,j} M_{i,j} M_{j,i}^T \right) \right\}$$

$$- \alpha_{i,j}^2 \sigma_{i,j} \left( F_{j,k} \sigma_{i,j} G_{i,j} U_i G_{j,k}^T \right) \times \sigma_{\min}(P_{j,k}, M_{j,k}^T M_{j,k})^{-1}$$

$$+ \alpha_{i,j} \left\|N_i^T Z_i^L N_i^T \right\|$$

(10)
is obtained recursively. For \( k = 1, \ldots, N \), one has (11), as shown at the bottom of the page.

One can obtain

$$
\Pi_k = \text{diag} \left[ Z_k^T - Z_k^T I + \text{diag} \left( \sum_{j=1}^{N} p_{jk} \pi_{i,j} G_{i,j} U_i G_{i,j}^T \right) \right]
$$

(12)

where

$$
Z_{k+1}^T := \mathbb{F} \left( Z_k^T - \gamma_{2i} N_f I + \gamma_{2i} N_f N_f^T \right)^{-1} \mathbb{F}^T - \gamma_{2i} M_f M_f^T
$$

and \( \gamma_{2i} \) is given by

$$
\gamma_{2i} = \tau_i \sigma_{\text{min}} \left( \text{diag} \left( \sum_{j=1}^{N} p_{jk} \pi_{i,j} G_{i,j} U_i G_{i,j}^T \right) \right) \times \sigma_{\text{max}} \left( M_f M_f^T \right)^{-1}
$$

with \( 0 < \tau_i < 1 \).

An advantage in solving lower and upper bounds based on optimization problems is that one can identify if there exists local minimum for (8) and (9), following studies performed in [34].

At this point, it has completed the definition of the augmented system (3). Bearing in mind that (12) provides the maximum influence of the uncertainties on the second moment of \( \psi_i \) in the System (3), it is proposed in the following a robust predictor \( \hat{z}_{i|k-1} \) for (3) based on the optimization problem

$$
\begin{align*}
\min_{\hat{z}_{i|k-1}} \max_{\mathcal{H}_k, \mathcal{U}_k} & \left\{ \left\| z_{i|k} - (F + \delta F) z_{i|k-1} \right\|_{W_i}^2 + \left\| z_{k|1} - \hat{z}_{i|k-1} \right\|_{W_i}^2 + \left\| y_k - (H_k + \delta H_k) z_{i|k-1} \right\|_{R_k}^2 \right\} \\
\end{align*}
$$

(13)

To obtain \( \hat{z}_{i|k} \) and \( Z_{i|k} \), from (13), this problem can be rewritten as

$$\begin{align*}
\min_{\delta A, \delta b} \max_{\delta A} & \left\{ \left\| x \right\|_{W}^2 + \left\| (A + \delta A) x \right\|_{W}^2 \right\} \\
\end{align*}
$$

(14)

$$
\hat{z}_{i|k} = H \Delta \left[ N_a N_b \right], \quad \left\| \Delta \right\| \leq 1
$$

(15)

where

$$
\begin{align*}
A & := \left[ \begin{array}{cc}
F & I \\
H_k & 0
\end{array} \right] ; & b & := \left[ \begin{array}{c}
F \hat{z}_{i|k-1} \\
H_k \hat{z}_{i|k-1} - y_k
\end{array} \right] \\
\delta A & := \left[ \begin{array}{cc}
\delta F & 0 \\
\delta H_k & 0
\end{array} \right] ; & \delta b & := \left[ \begin{array}{c}
\delta F \hat{z}_{i|k-1} \\
\delta H_k \hat{z}_{i|k-1} - \hat{z}_{i|k-1}
\end{array} \right] \\
V & := \left[ \begin{array}{cc}
\hat{Z}_{i|k}^T & 0 \\
0 & 0
\end{array} \right] ; & W & := \left[ \begin{array}{cc}
\Pi_i^{-1} & 0 \\
0 & 0
\end{array} \right] \\
H & := \left[ \begin{array}{cc}
M_f & 0 \\
0 & M_h
\end{array} \right] ; & N_i & := \left[ \begin{array}{cc}
N_f & 0 \\
0 & N_h
\end{array} \right] \\
N_a & := \left[ \begin{array}{cc}
N_f & 0 \\
0 & N_h
\end{array} \right] ; & \Delta & := \left[ \begin{array}{c}
\hat{z}_{i|k-1} - z_i \\
0
\end{array} \right]
\end{align*}
$$

According to [26], the solution of the optimization problem (14)–(15) is given by

$$
\hat{z} = (V + A^T W A)^{-1} (A^T W b + \lambda N_a N_b) \hat{z}_{i|k-1}
$$

(16)

where the modified weighting matrices \( V \) and \( W \) are defined by

$$
\begin{align*}
V & := V + \lambda N_a N_a ; \\
W & := W + W H (\lambda I - H^T W H)^{-1} H^T W
\end{align*}
$$

and \( \lambda \) is a nonnegative scalar parameter which satisfies \( \lambda \geq \| H^T W H \| \). With these definitions, one can propose an algorithm to compute \( \hat{z}_{i+1|k} \), following the framework developed in [25].

**Recursive Robust Predictor for DMJLS**

**Step 0:** (Initial conditions): \( \hat{z}_{i|k-1} := P_i ; \quad \hat{z}_{i|k-1} := 0 \).

**Step 1:** Compute \( R_i, Z_i^T, Z_i^T, \) and \( \Pi_i \) through (6), (8), (9), and (10), respectively.

\[
\begin{bmatrix}
Z_i^T & - \sum_{j=1}^{N} p_{jk} F_j Z^T_{i,j} \Sigma_i \Sigma_i \Sigma_i \Sigma_i \Sigma_i \Sigma_i \Sigma_i \Sigma_i \Sigma_i \Sigma_i \Sigma_i \Sigma_i \Sigma_i \Sigma_i \Sigma_i \Sigma_i \Sigma_i \Sigma_i \Sigma_i \Sigma_i \Sigma_i
\end{bmatrix}
\begin{bmatrix}
\Sigma_i I - N_i^T N_i^T & 0 & \cdots & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
> 0
\end{bmatrix}
\]
Step 2: If $M^T_i = 0$ and $M^h = 0$, then $\lambda_i = 0$. Otherwise, it is chosen as
\[
\lambda_i = (1 + \rho_{i+1}) \times \begin{bmatrix} M^T_i & 0 \\ 0 & M^h \end{bmatrix}^{-1} \begin{bmatrix} \Pi_{i+1}^{-1} & 0 \\ 0 & R_{i+1}^{-1} \end{bmatrix} \begin{bmatrix} M^T_i & 0 \\ 0 & M^h \end{bmatrix}
\]  
(17)
where $\rho_{i+1} > 0$ and the parameters $\{R_i, \Pi_i\}$ are changed to the corrected parameters
\[
\tilde{R}_i := R_i - \lambda_i^{-1} M^h M^T_i, \quad \tilde{\Pi}_i := \Pi_i - \lambda_i^{-1} M^T_i M_i^h.
\]  
(18)

Step 3: Actualize $\{\tilde{Z}_i, \tilde{z}_i\}$ with $\{\tilde{Z}_{i+1}, \tilde{z}_{i+1}\}$ through the following recursive equations
\[
\tilde{Z}_{i+1} := \tilde{Z}_i + \mathcal{F} \tilde{Z}_i \tilde{F} \tilde{z}_{i+1}
\]  
(19)
\[
\tilde{z}_{i+1} := \mathcal{F} \tilde{z}_i + \mathcal{F} \tilde{Z}_i \tilde{F} \tilde{z}_{i+1}
\]  
(20)

where $\tilde{R}_i := \begin{bmatrix} \tilde{R}_i & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \Pi_{i+1}^{-1} & 0 \\ 0 & R_{i+1}^{-1} \end{bmatrix} \begin{bmatrix} \Pi_{i+1} & 0 \\ 0 & R_{i+1} \end{bmatrix}$ and
\[N = \lambda_i \left( \begin{bmatrix} N_f & N_f \end{bmatrix} + \begin{bmatrix} N_h & N_h \end{bmatrix} \right) \]  

Step 4: From $\tilde{z}_{i+1}$, compute $\tilde{x}_{i+1} = [\tilde{z}_{i+1}^T, \tilde{y}_{i+1}^T]^T$.

If the uncertainties of the robust predictor are canceled ($M^T_i = M^h = N_f = N_h = 0$), it reduces to the nominal predictor developed in [11]. For the case with no jumps ($N = 1$), it resembles to the robust estimator developed in [25] (when the singular system of this reference is considered in the standard state-space form).

Remark 1: In order to guarantee the stability of the stationary robust predictor proposed, some features are assumed: System (1) is mean square stable (MSS), the Markov chain is ergodic and $\tilde{R}_{i+1} = \tilde{R}$ > 0 (and with the condition (17) satisfied, the positiveness of $\tilde{R}_{i+1} = \tilde{R}$ is also assured). For $\alpha_{i,j} = \kappa_{i,j}(k) = \kappa_{j,i}(k) = \alpha_{i,j}$, $\beta_{i,j}(k) = \beta_{j,i}(k)$ fixed and for all model parameters constant, it is considered the following algebraic Riccati Equation:
\[
\dot{Z} := \tilde{\Pi} + \mathcal{F} \tilde{Z} \tilde{F}^T - \mathcal{F} \mathcal{Z} \begin{bmatrix} \Pi_{i+1} & 0 \\ 0 & R_{i+1} \end{bmatrix} \begin{bmatrix} \Pi_{i+1}^{-1} & 0 \\ 0 & R_{i+1}^{-1} \end{bmatrix} \begin{bmatrix} \Pi_{i+1} & 0 \\ 0 & R_{i+1} \end{bmatrix} \dot{Z} \begin{bmatrix} \Pi_{i+1} & 0 \\ 0 & R_{i+1} \end{bmatrix}^T
\]  
(21)
with $\dot{\tilde{R}} := \begin{bmatrix} \tilde{R}_i & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \Pi_{i+1} & 0 \\ 0 & R_{i+1} \end{bmatrix} \begin{bmatrix} \Pi_{i+1}^{-1} & 0 \\ 0 & R_{i+1}^{-1} \end{bmatrix} \begin{bmatrix} \Pi_{i+1} & 0 \\ 0 & R_{i+1} \end{bmatrix}$.

There exists a unique positive-semidefinite solution $\dot{Z}$ (with $i \to \infty$) for (21), following the guidelines defined in [11], and
\[
\sigma_{\dot{Z}} \left( \mathcal{F} - \mathcal{F} \mathcal{Z} \begin{bmatrix} \Pi_{i+1} & 0 \\ 0 & R_{i+1} \end{bmatrix} \begin{bmatrix} \Pi_{i+1}^{-1} & 0 \\ 0 & R_{i+1}^{-1} \end{bmatrix} \begin{bmatrix} \Pi_{i+1} & 0 \\ 0 & R_{i+1} \end{bmatrix} \right) < 1
\]  
where $\sigma_{\dot{Z}}(\cdot)$ denotes spectral radius of the predictor with stationary gain. The stability of this estimator is easily checked thanks to the characteristic of the parameter $\dot{\lambda}_i$, it depends on only known parameters of the DMJLS.

The robust recursive filter presented in the next algorithm, shown through (23)–(26), is deduced based on the optimization problem
\[
\min_{\dot{z}_{i+1}, \dot{z}_{i+1}} \max_{\dot{y}_{i+1}, \dot{y}_{i+1}} \left\{ \| \dot{z}_{i+1} - (\mathcal{F} + \dot{\mathcal{F}}) \dot{z}_i \|_{\Pi_i}^2 + \| \dot{y}_i - \dot{y}_{i+1} \|_{\dot{R}_{i+1}}^2 \right\}
+ \min_{\dot{y}_{i+1}, \dot{y}_{i+1}} \max_{\dot{z}_{i+1}, \dot{z}_{i+1}} \left\{ \| \dot{z}_i - (\mathcal{H} + \dot{\mathcal{H}}) \dot{z}_{i+1} \|_{\dot{R}_{i+1}}^2 \right\}.
\]  
(22)

Following the guidelines aforementioned, (22) can be rewritten in the same way it was written (14), whose solution is given by (16). The robust filters proposed in this technical note, for $N = 1$ and without uncertainties, reduce to the standard Kalman filter [35].

**Recursive Robust Filter for DMJLS**

Step 0: (Initial conditions): $\dot{R}_0 := R_0 - \lambda_i^{-1} M^h M^T_i$, $\dot{Z}_0 := \begin{bmatrix} P_0^{-1} + \lambda_i^{-1} M^h \dot{R}_0^{-1} M^T_{i+1} \\ \dot{z}_0 \end{bmatrix}$, $\dot{z}_0 := \begin{bmatrix} \dot{y}_0 \\ \dot{y}_0 \end{bmatrix}$.

Step 1: Compute $R_{i+1}, Z_{i+1}^f, Z_{i+1}^g$ and $\Pi_i$ through (6), (8), (9), and (10), respectively.

Step 2: If $M^T_i = 0$ and $M^h = 0$, then $\lambda_i = 0$. Otherwise, it is chosen
\[
\lambda_i = (1 + \rho_{i+1}) \begin{bmatrix} M^T_i & 0 \\ 0 & M^h \end{bmatrix} \begin{bmatrix} \Pi_{i+1}^{-1} & 0 \\ 0 & R_{i+1}^{-1} \end{bmatrix} \begin{bmatrix} M^T_i & 0 \\ 0 & M^h \end{bmatrix}
\]  
(23)
where $\rho_{i+1} > 0$ and the parameters $\{R_i, \Pi_i\}$ are changed to the corrected parameters
\[
\dot{R}_{i+1} := R_{i+1} - \lambda_i^{-1} M^h M^T_{i+1}, \quad \dot{\Pi}_{i+1} := \Pi_{i+1} - \lambda_i^{-1} M^T_{i+1} M^h.
\]  
(24)

Step 3: Actualize $\{\dot{Z}_{i+1}, \dot{z}_{i+1}\}$ with $\{\dot{Z}_{i+1}, \dot{z}_{i+1}\}$ through the recursive equations shown in (25)–(26), shown at the bottom of the page, where $\dot{\Pi}_{i+1} := \begin{bmatrix} \dot{\Pi}_{i+1} & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \mathcal{F} & \mathcal{F} \end{bmatrix} \begin{bmatrix} \mathcal{F} \end{bmatrix}^T$.

Step 4: From $\dot{z}_{i+1}$, compute $\dot{x}_{i+1} = \sum_{i=1}^{N} \dot{z}_{i+1}^i$.

\[
\dot{Z}_{i+1} := \begin{bmatrix} T^T \\ 0 \end{bmatrix} \dot{\Pi}_{i+1} \begin{bmatrix} T \\ 0 \end{bmatrix} + \begin{bmatrix} \mathcal{H}_{i+1}^T \\ 0 \end{bmatrix} \begin{bmatrix} \dot{R}_{i+1}^{-1} & 0 \\ 0 & \mathcal{H}_{i+1} \end{bmatrix} \begin{bmatrix} \mathcal{H}_{i+1} \end{bmatrix} \begin{bmatrix} \mathcal{H}_{i+1} \end{bmatrix}^{-1}
\]  
(25)
\[
\dot{z}_{i+1} := \dot{Z}_{i+1} - \dot{Z}_{i+1} \begin{bmatrix} T^T \\ 0 \end{bmatrix} \dot{\Pi}_{i+1} \begin{bmatrix} T \\ 0 \end{bmatrix} + \begin{bmatrix} \mathcal{H}_{i+1}^T \dot{R}_{i+1}^{-1} \mathcal{H}_{i+1}^T \end{bmatrix} \begin{bmatrix} y_{i+1}^i \\ 0 \end{bmatrix}
\]  
(26)
IV. NUMERICAL EXAMPLE

In the following, two comparative studies are presented to show the effectiveness of the robust predictor (17)–(20). Both studies were obtained based on the System (1)–(2) with two Markovian states, transition probability matrix $P = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$, and parameter matrices defined as

$$
F_1 = \begin{bmatrix} 0.7 & 0 \\ 0.1 & 0.2 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0.6 & 0 \\ 0.1 & 0.2 \end{bmatrix},
$$

$$
G_1 = G_2 = \begin{bmatrix} 0.8731 & 0 \\ 0 & 0.2089 \end{bmatrix},
$$

$$
M_1^f = \begin{bmatrix} 0.13 & 0 \\ 0 & 0.13 \end{bmatrix}, \quad M_2^f = \begin{bmatrix} 0.13 & 0 \\ 0 & 0.13 \end{bmatrix},
$$

$$
M_1^u = \begin{bmatrix} 0.2 & 0 \\ 0.1 & 0.5 \end{bmatrix}, \quad M_2^u = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.5 \end{bmatrix},
$$

$$
H_1 = H_2 = \begin{bmatrix} 0.1 & 0.0 \\ 0 & 0.1 \end{bmatrix}, \quad D_1 = D_2 = -0.008,
$$

$$
M_1^h = M_2^h = \begin{bmatrix} 0.39 & 0 \\ 1 & 0 \end{bmatrix}, \quad N_1^h = N_2^h = \begin{bmatrix} 1.3 & 0 \\ 0 & 5 \end{bmatrix}.
$$

First, a comparative study between the standard Kalman filter (SKF) developed in [11] for DMJLS and the robust predictor proposed is shown in Fig. 2. The square root of the mean square error (rms) estimator given in [11] was simulated for the system with and without uncertainties, whereas the robust predictor was simulated for the system with uncertainties. The curves were obtained from $i = 0, \ldots, 100$ with the values of $\Theta_i$ generated randomly. The initial condition $x_0$ was considered Gaussian with mean $[0.169 \ 0.295]^T$ and variance $[0.0384 \ 0.0578 \ 0.0578 \ 0.870]$, $\Theta_i \in \{1,2\}$, $\tau_i$ and $\theta_i$ are independent sequences of noises, $\tau_1(0) = 0.95$ and $\tau_2(0) = 0.95$. The parameters $\rho_{ij}$ and $\tau_i$ were settled as 2 and 0.1, respectively. Notice that the parameter $\lambda_i$ changes for each recursive step of 400 Monte Carlo simulations. It is important to point out also that if $\rho_{1i} > 0$ in (17), the stability of the robust predictor is always guaranteed. The tuning of the variance $\Pi_i$ in (12) influences only the robust predictor performance, the stability is always guaranteed. Notice in Fig. 2 the advantage of using the robust predictor when the DMJLS is subject to uncertainties.

Second, a comparative study between the robust Markovian estimator proposed in [16] and the robust predictor developed is shown in Fig. 3. The predictor of [16] was deduced for polytopic uncertainties and calculated exclusively through LMIs. According to Remark 7.1 of [36, p. 265], polytopic and norm bounded uncertainties can be equivalent in some cases. One of these cases is when the norm bounded uncertainties are diagonal. In this form of representation, both uncertainties result in a polyhedral convex set. Four scalar numerical models considered in [16] were taken into account here. Our estimator outperforms the estimator given in [16] in three cases: a, c, and d; and under-performs [16] in the case b. Both estimators are optimal in some sense, however the approach proposed in this technical note is more useful for online applications.

V. CONCLUSION

In this technical note, robust Kalman-type filters were developed for DMJLS. The main feature of these filters is the guarantee of stability in online applications. Furthermore, provided that the variances of the augmented model of the DMJLS system are calculated, the optimal filters can be obtained following the guidelines proposed in [25]. In review, this note shows that after some algebra, it is possible to apply recursive robust techniques to estimate DMJLS, even when they are time-varying. For future works, one intends to deduce recursive robust estimators for this class of systems which do not depend on any tuning parameter.

REFERENCES


has distinct eigenvalues. Unfortunately, this fact no longer holds when
parameters. The reason for this is that the diagonal Padé approximation,
under the assumption that the continuous-time system matrix of
preserves the Jordan structure of a matrix. However, the analysis of
existence of nontrivial Jordan blocks, and the purpose of this technical
note is to extend the results of [1] to the case of nontrivial Jordan blocks.

Polynomial Lyapunov functions are known to be nonconservative in
analysis of stability under arbitrary switching for polytopic and
switched systems, when compared to quadratic Lyapunov functions
properties of the system matrix $A_i$ that is shared under discretization
by discretization is discussed in [1]. Recall that the

Extensions of “Padé Discretization for Linear Systems With Polyhedral Lyapunov Functions”
for Generalized Jordan Structures

Surya Shravan Kumar Sajja, Francesco Rossi, Patrizio Colaneri, and
Robert Shorten

Abstract—Recently, we showed that certain types of polyhedral Lyapunov functions for linear time-invariant systems are preserved by diagonal Padé approximations, under the assumption that the continuous-time system matrix $A_i$ has distinct eigenvalues [1]. This result follows by making explicit use of the fact that the diagonal Padé approximation preserves the Jordan structure of a matrix $A_i$, if the matrix has distinct eigenvalues. Unfortunately, this fact no longer holds when $A_i$ has nontrivial Jordan blocks, and the purpose of this technical note is to extend the results of [1] to the case of nontrivial Jordan blocks.

Polynomial Lyapunov functions are known to be nonconservative in the
analysis of stability under arbitrary switching for polytopic and
switched systems, when compared to quadratic Lyapunov functions [2]. The motivation for wondering whether there exists a polyhedral LF that is shared under discretization is discussed in [1]. Recall that the

I. INTRODUCTION

Recently, we showed that certain types of polyhedral Lyapunov functions for linear time-invariant systems are preserved by diagonal Padé approximations, under the assumption that the continuous-time system matrix $A_i$ has distinct eigenvalues [1]. This result follows by making explicit use of the fact that the diagonal Padé approximation preserves the Jordan structure of a matrix $A_i$, if the matrix has distinct eigenvalues. Unfortunately, this fact no longer holds when $A_i$ has nontrivial Jordan blocks, and the purpose of this technical note is to extend the results of [1] to the case of nontrivial Jordan blocks.

Polynomial Lyapunov functions are known to be nonconservative in the
analysis of stability under arbitrary switching for polytopic and
switched systems, when compared to quadratic Lyapunov functions [2]. The motivation for wondering whether there exists a polyhedral LF that is shared under discretization is discussed in [1]. Recall that the