Scalar field description for parametric models of dark energy
I. INTRODUCTION

Cosmological models with cold dark matter plus a $\Lambda$ term ($\Lambda$CDM) may explain most of the current astronomical observations (see, e.g., [1] for some recent reviews). However, from the theoretical viewpoint it is really difficult to reconcile the small value required by observations ($\approx 10^{-10}$ erg/cm$^3$) with estimates from quantum field theories ranging from 50–120 orders of magnitude larger, as well as to explain why this is exactly the right value that is just beginning to dominate the energy density of the Universe today.

These issues make a complete cancellation of $\Lambda$ (from some unknown symmetry of Nature) seem a plausible possibility and have also motivated a number of alternative explanations for the cosmic acceleration (for some of these alternative scenarios, see [2]). One of these possibilities, possibly the next simplest approach toward an accelerating model for the Universe, is to work with the idea that the dark energy component is due to a minimally coupled scalar field $\phi$ which has not yet reached its ground state and whose current dynamics is basically determined by its potential energy $V(\phi)$ [3]. Clearly, however, such a procedure cannot provide a model-independent parameter space to be compared with the observational data.

Another way, widely explored in the literature, is to build a phenomenological functional form for the dark energy equation of state (EoS), i.e., the ratio of its pressure to its energy density, $w \equiv p/\rho$, in terms of its current value $w_0$ and of its time-dependence $w' = dw/dz|_{z=0}$, and study its cosmological consequences as well as possible constraints on its behavior from observations. Usually, these parameterizations have not only the standard $\Lambda$CDM scenario ($w_0 = -1; w'_0 = 0$) but also the so-called $w$CDM model ($w'_0 = 0$) as particular cases, so that constraints on its parameters may provide more accurate consistency checks to the original models.

Examples of some EoS parameterizations recently explored are (see also [4–6]):

$$w(z) = \begin{cases} w_0 + w'_0 z & [7] \\ w_0 + w'_0 z/(1 + z) & [8] \\ w_0 - w'_0 \ln(1 + z) & [9] \end{cases}$$

(1)

An interesting aspect worth mentioning is that it is difficult to obtain the above parameterizations from usual scalar field dynamics since they are not limited functions, i.e., the EoS parameter does not lie in the interval $w \in [-1, 1]$. In other words, this amounts to saying that when extended to the entire history of the Universe, $z \in [-1, \infty)$, the three parameterizations above are divergent functions of the redshift (see also [5] for a discussion). It is worth mentioning that the class of $w(z)$ parameterizations above can be expressed as a single and generalized EoS function, i.e.,

$$w(z) = w_0 - w'_0 (\frac{1+z}{\beta})^{\beta-1},$$

where the parameter $\beta$ takes, respectively, the values $\pm 1$ and 0 [6].

In Ref. [7] we investigated some cosmological consequences of a new phenomenological parameterization:

$$w(z) = w_0 + w'_0 \frac{z(1+z)}{1+z^2}.$$  

(2)

This parameterization has the same linear behavior at low redshifts presented by the parameterizations discussed above but with the advantage of being a limited function of $z$ throughout the entire history of the Universe (see also [8] for a recent analysis of a coupled quintessence model driven by a dark energy component parameterized by (2)).
Our goal in this paper is twofold. First, to extend our previous analysis [7] to the case in which dark matter and dark energy described by EoS (2) are coupled following a coupling term of the type $\rho_{dm}/\rho_s \propto (1+z)^{\alpha}$ [9] and to place constraints on the parameters $w_0$, $w'_0$ and $\alpha$ from current observational data to check if there are some class of dark energy and if models with interaction are favored observationally. We use type Ia supernovae (SNe Ia) observations from Union2 sample [10]. Along with the SNe Ia data, and to help break the degeneracy between the dark energy parameters we also use measurements of the baryonic acoustic oscillation (BAO) peak at $z = 0.2$, $0.35$ and $0.6$ [11–13] and the current estimate of the CMB shift parameter of Ref. [14]. Second, to derive a scalar field description for the dark component whose EoS parameter is given by Eq. (2). To that end, we use the theoretical method of constructing the dark energy potential $V(\phi)$ directly from the effective EoS, as developed in Ref. [15]. We generalize this algorithm to include the so-called phantom case and we also apply it to two classes of models (uncoupled and coupled dark energy scenarios).

II. MODELS

In the following we restrict our analysis to a homogeneous, isotropic, spatially flat cosmologies described by the Friedmann-Robertson-Walker flat line element, $dx^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)$, where $a(t)$ is the scale factor and we have set the speed of light $c = 1$. We will focus on two distinct scenarios: a coupled and uncoupled dark energy component described by (2).

Case I: uncoupled dark energy—In this case, the dark energy density satisfies the equation $\dot{\rho}_s + 3H[1 + w(z)])\rho_s = 0$, where $w(z)$ is given by parameterization (2). Thus, the DE density evolves as

$$f_1(z) = \frac{\rho_s}{\rho_{x,0}} = (1+z)^{3(1+w_0)(1+z)^3 w'_0/2},$$

and the Friedmann equation for a dark matter/dark energy dominated universe reads

$$H^2 = H_0^2[\Omega_{m,0}(1+z)^3 + (1 - \Omega_{m,0})f_1(z)],$$

where $\Omega_{m,0} = \rho_{m,0}/\rho_{c,0}$ ($\rho_{c,0} = 3H_0^2/8\pi G$) is density parameter of the matter component.

Case II: coupled dark energy—Now, we will introduce a coupling between dark energy and dark matter, so that the energy conservation law is written as $\dot{\rho}_{dm} + 3H\rho_{dm} = -\dot{\rho}_s - 3H[1 + w(z)]\rho_s$, where $\rho_{dm}$ is dark matter density. We will assume that the dark fluids are related by $\rho_{dm}/\rho_s = \rho_{dm,0}/\rho_{s,0}(1+z)^{\alpha}$, where $\alpha$ is a constant [9]. Note that a negative value of $\alpha$ implies an early dominance of the dark energy on the dark matter. Combining the two latter equations and replacing $w(z)$, as given in (2), we obtain

$$f_2(z) = (1+z)^{3(1+w_0)}\left[\Omega_{x,0} + \Omega_{dm,0}(1+z)^{\alpha}\right]^{(3w_0/\alpha)+1} \times \exp\left[\int_0^z \frac{3\Omega_{x,0}w'_0x^2 dx}{(1+x^2)(\Omega_{x,0} + \Omega_{dm,0}(1+z)^{\alpha})}\right].$$

The expression for a constant EoS is obtained by making $w'_0 = 0$ in (5) and the $\Lambda$CDM case stands for $w_0 = -1$, $w'_0 = 0$ and $\alpha = 3$. The Friedmann equation for this interaction model (with the baryonic component conserved separately) can be written as

$$H^2 = H_0^2[\Omega_{b,0}(1+z)^3 + \Omega_{dm,0}(1+z)^{\alpha}] + (1 - \Omega_{dm,0} - \Omega_{b,0})f_2(z).$$

III. CONSTRAINTS ON THE DARK ENERGY MODELS

Before proceeding to the observational analyses on the $w_0 - w'_0$ parametric plane, it is worth mentioning that the parameters $w_0$ and $w'_0$ of (2) are subject to the following constraints: $-1 \leq w_0 - 0.21w'_0$ and $w_0 + 1.21w'_0 \leq 1$ (if $w_0 < 0$) and $-1 \leq w_0 + 1.21w'_0$ and $w_0 - 0.21w'_0 \leq 1$ (if $w_0 < 0$) for a quintessence-like behavior and $w_0 - 1 = (-1+w_0)/1.21$ (if $w_0 < 0$) and $w'_0 > (1+w_0)/0.21$ (if $w'_0 < 0$) for phantom fields. These bounds allow us to divide the parametric plane $(w_0 - w'_0)$ in defined regions associated to distinct classes of dark energy models that can be confronted with current observational data. These regions are shown in the Figs. 1(c) and 1(d), where the area of early dark energy dominance corresponds to the region where the constraint $w_0 + w'_0 < 0$, required to ensure $\rho_m(z) > \rho_s(z)$ at $z \gg 1$ is violated. The blank regions indicate models that at some point of the cosmic evolution, $z \in [-1, \infty)$, have switched or will switch from quintessence to phantom behaviors or vice-versa.

In order to discuss the current observational constraints on $w_0$, $w'_0$ and $\alpha$, we use the Union2 SNe Ia sample of Ref. [10]. The Union2 sample is an update of the original Union compilation. It comprises 557 data points including recent large samples from other surveys and uses SALT2 for SN Ia light-curve fitting [16]. Along with the SNe Ia data, and to diminish the degeneracy between the dark energy parameters $\Omega_{m,0}$, $\alpha$, $w_0$ and $w'_0$, we also use the results of current BAO and CMB experiments. For the BAO measurements, we use the three estimates of the BAO parameter $\mathcal{A}(z) = D_V\sqrt{\Omega_m H_0^2}$ at $z = 0.2$, $0.35$ and $0.6$, as given in Table 2 of Ref. [13]. In this latter expression, $D_V = [r_0(z_{BAO})^2 H(z_{BAO})]^{1/3}$ is the so-called dilation scale, defined in terms of the dimensionless comoving distance $r$. For the CMB, we use only the measurement of the CMB shift parameter $\mathcal{R} = \Omega_m H_0^2 r(z_{CMB}) = 1.725 \pm 0.018$, where $z_{CMB} = 1089$ [14]. In our analyses, we minimize the function $x^2 = \chi_{\Lambda\text{CDM}}^2 + \chi_{\text{BAO}}^2 + \chi_{\text{CMB}}^2$, which takes into account all the data sets.
FIG. 1. Contours of $\Delta \chi^2 = 2.30$ and $6.17$. Panels (a) and (b) show the $w_0 - \alpha$ parametric space for interacting models with constant ($w_0' = 0$) and variable EoS, respectively. Panels (c) and (d) show the $w_0 - w_0'$ parametric space for uncoupled and coupled cases, respectively.

mentioned above and marginalize over the present value of the Hubble parameter $H_0$. Also, we fix $\Omega_{b,0} = 0.0416$ from WMAP results [17] (which is also in good agreement with the bounds on the baryonic component derived from primordial nucleosynthesis [18]).

Figure 1 shows the results of our statistical analyses at 68.3% and 95.4% confidence levels. The $\alpha - w_0$ parameter space is shown in the top graphs for a constant EoS (Panel a) and for the parameterization (2) (Panel b). The $w_0 - w_0'$ parameter space is shown in the bottom plots for the uncoupled case (Panel c) and for the coupled case with (2) (Panel d). We note that no dark energy behavior is preferred or ruled out by observations, although the largest portion of the confidence contours lies into the blank region, which indicates a possible crossing of the so-called phantom divide line at some point of the cosmic evolution (see [19] for a discussion).

In Table I we summarize the results of our statistical analyses. Since the number of free parameters is the same for all cases, we compare the models by using the $\chi^2_{\text{min}}$ values. We note that models with interaction in the dark sector seem to provide a better fit to the data. For the sake of comparison, we also performed the analyses for the so-called CPL parameterization, $w(z) = w_0 + w_0' z/(1 + z)$ [20]. We have found very similar values to those shown in Table I. This means that, although different from the theoretical viewpoint—CPL parameterization blows up exponentially in the future as $z \to -1$ for $w_0 > 0$ whereas Parameterization (2) is a limited function of $z \forall z \in [-1, \infty]$—both parameterizations provide very similar description for the current observational data.

### IV. SCALAR FIELD DESCRIPTION

For a scalar field, the energy density and pressure are given by $\rho = \epsilon \frac{1}{2} \dot{\phi}^2 + V(\phi)$ and $p = \epsilon \frac{1}{2} \dot{\phi}^2 - V(\phi)$, where $\epsilon = \pm 1$ stands for canonic (quintessence) ($-1 \leq w \leq 1$) [3] and noncanonic (phantom) fields ($w < -1$)

<table>
<thead>
<tr>
<th>Table I.</th>
<th>The results of our analyses for the uncoupled case with parameterization (2) (M1) and for coupled dark energy models with: a constant EoS (M2) and parameterization (2) (M3). The error bars correspond to 68.3% C.L.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi^2_{\text{min}}$</td>
<td>$w_0$</td>
</tr>
<tr>
<td>M1......</td>
<td>545.02</td>
</tr>
<tr>
<td>M2......</td>
<td>544.77</td>
</tr>
<tr>
<td>M3......</td>
<td>544.61</td>
</tr>
</tbody>
</table>
From the above equations, we obtain

\[ \dot{\phi}^2 = \frac{1 + w}{e} \rho_x \quad \text{and} \quad V(\phi) = \frac{1}{2} (1 - w) \rho_x, \]  

(7)

or still, in terms of \( z \),

\[ \dot{\phi} = \frac{d\phi}{dz} \dot{z} = - \frac{d\phi}{dz} (1 + z) H(z), \]  

(8a)

so that,

\[ \frac{d\phi}{dz} = \pm \frac{1}{(1 + z) H(z)} \sqrt{\frac{1 + w}{e} \rho_x}, \]  

(8b)

where the negative (positive) signs stands to \( \dot{\phi} > 0 \) \( (\dot{\phi} < 0) \). Here, we adopt the negative sign.

By defining \( \phi = \sqrt{8 \pi G/3} \phi \) and \( V = V/\rho_{c,0} \) and taking into account that \( (1 + w)/e = |1 + w| \), we have

\[ \Delta \phi = \phi - \phi_0 = - \int_0^z \frac{1}{(1 + z) \eta(z)} \sqrt{1 + w(z)} |\Omega_{x,0} f(z)| \]  

(9a)

and

\[ \tilde{V}(\phi) = \frac{1}{2} [1 - w(z)] \Omega_{x,0} f(z), \]  

(9b)

where \( \eta(z) = H(z)/H_0 \) and \( f(z) = \rho_x/\rho_{c,0} \) stands for the time-dependent part of the dark energy density. Note also that Eqs. (9a) and (9b) are valid for both quintessence and phantom fields.

**Case I: uncoupled dark energy**—By combining numerically Eqs. (2)–(4), (9a), and (9b), and taking into account

FIG. 2. Scalar field description for Case I (uncoupled) for three selected points in quintessence (Panels a and b) and phantom (Panels c and d) regions.

FIG. 3. Scalar field description for Case II for three selected values of \( \alpha \) and some values in both quintessence (Panels a and b) and phantom (Panels c and d) regions. The uncoupled case also is shown (solid lines).
the above constraints, we show in Figs. 2(a) and 2(c) the resulting potential $\tilde{V}(\dot{\phi})$ for quintessence and phantom regimes, respectively. Figs. 2(b) and 2(d) show the evolution of the dark energy field as function of the redshift. The selected points used to plot the curves belong to quintessence and phantom families with $w(z) > 1 = w_0 + w'_0 = -0.6$ and $w(z) > 1 = w_0 + w'_0 = -1.2$ and follow the constraints given in Sec. III. We note that, in contrast to the uncoupled case (full lines), as we can see, in the case with interaction the scalar field rolls more smoothly until a minimum of its potential. The main difference between the uncoupled and coupled cases occurs in the phantom regime. We also have performed a joint statistical analysis involving some of the most recent cosmological measurements of SNe Ia, BAO peak and CMB shift parameter. From a pure observational perspective, we have shown that both quintessence and phantom behaviors are acceptable regimes. In agreement with recent analyses, it has been shown that the largest portion of the confidence contours arising from these observations lies in the region of models that have crossed or will eventually cross the so-called phantom divide line at some point of the cosmic evolution.

ACKNOWLEDGMENTS

This work is supported by CNPq, FAPERJ, FAPERN and FAPESP.


