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A Levinson Algorithm Based on an Isometric Transformation of Durbin's

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***Abstract*—Starting from Durbin algorithm in polynomial space with an inner product defined by the signal autocorrelation matrix, an isometric transformation is defined that maps this vector space into another one where Levinson algorithm is performed. Alternatively, for iterative algorithms such as discrete all-pole (DAP), an efficient implementation of a Gohberg-Semencul (GS) relation is developed for the inversion of the autocorrelation matrix which considers its centrosymmetry. In the solution of the autocorrelation equations, Levinson algorithm is found to be less complex operationally than the procedures based on GS inversion for up to a minimum of five iterations at various linear prediction (LP) orders.**

***Index Terms*—Levinson algorithm, Durbin algorithm, linear prediction, LP analysis, AR models, discrete all-pole (DAP)**

I. Introduction

THE most widespread technique used in current speech coding and analysis is linear prediction. It involves a set of normal equations that can be efficiently solved by Durbin algorithm [1] when the coefficient matrix is an autocorrelation matrix. This algorithm may be set in polynomial space where its solution becomes very appealing for its elegance and simplicity [2].

Actually, the normal equations may be represented in polynomial space even when the coefficient matrix is not Toeplitz [2], in which case we have the covariance method of LP. Also, for a symmetric Toeplitz matrix like the autocorrelation matrix, other efficient algorithms may be used for solving the normal equations such as the Schur or Le Roux-Gueguen algorithm, already derived in polynomial space [3].

Normal equations arise when the speech spectrum is fit by a continuous spectral model. However, a discrete spectrum is a more exact model for voiced speech that is used in sinusoidal speech coding [4]. In this situation, the equations for the model coefficients are actually nonlinear. This difficulty is usually circumvented by interpolating the discrete spectrum to obtain a continuous one which is then fit by LP models [4].

Eventually, an iterative algorithm has been devised for obtaining the model coefficients as the solution to a modified set of linear equations with a different full right-hand side vector in each iteration. This method is called discrete all-pole (DAP) [5].

Unfortunately, the full set of equations for DAP modeling cannot be solved by Durbin algorithm. However, Levinson algorithm solves them since it was originally proposed for Wiener filtering and prediction, where they also arise [6].

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This work casts Levinson algorithm in a polynomial space which is the image of a linear transformation from the polynomial space where Durbin algorithm takes place. Its simplicity derives from the isometry of the transformation as explained in Section III after reviewing Durbin algorithm and establishing notation in Section II. Since the coefficient matrix in DAP method is the same for all iterations, its inversion is explored in Section IV by means of a Gohberg-Semencul (GS) relation [7], for which a more efficient implementation is proposed based on the centrosymmetry of the inverse matrix. The arithmetic operation count for the algorithms is analyzed and compared in Section V and, in conclusion, some remarks to the applicability of the algorithms are drawn.

II. Review of Durbin algorithm and notation

The set of normal equations arising in autoregressive analysis may be cast in matrix form as

$$\mathbf{R}\mathbf{a} = \boldsymbol{\delta}_0, \tag{1}$$

where $\mathbf{R}$ is the $(p+1)\times(p+1)$ autocorrelation matrix for LP analysis of order p and $\boldsymbol{\delta}_0 = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix}^T$ is a $(p+1)\times 1$ column vector. The augmented autocorrelation matrix $\mathbf{R}$ induces an inner product over the vector space of polynomials by means of the inner product of monomial polynomials [2] defined by

$$\langle z^{-i}, z^{-j} \rangle \triangleq r_{ij}. \tag{2}$$

for $i, j \in \{0, 1, \dots, p\}$. Using polynomials, the last p normal equations in Eq. (1) may be written as

$$\langle C(z), z^{-j} \rangle = 0, \quad j = 1, 2, \dots, p \tag{3}$$

within a multiplying factor, where

$$C(z) = C_p(z) = 1 + \sum_{i=1}^{p} c_{pi} z^{-i} \tag{4}$$

is the inverse LP filter that satisfies the normal equations (3). The algorithm constructs an orthogonal vector basis to the polynomial space in iterations of increasing degree where basis polynomial

$$B_m(z) = z^{-(m+1)} + \sum_{i=1}^{m} b_{mi} z^{-i} \tag{5}$$

is the mth-order backward prediction inverse filter. Let the square norms of $B_m(z)$ be $\beta_m = \langle B_m(z), B_m(z) \rangle$ for $m = 0, 1, \dots, p$, respectively, according to the polynomial inner product.

The process starts trivially with the zeroth-order inverse filters $C_0(z) = 1$ and $B_0(z) = z^{-1}$. In solving for the mth-order predictor, the value of parameter κ_m is chosen to be

$$\kappa_m = -\frac{\langle C_{m-1}(z), z^{-m}\rangle}{\beta_{m-1}} \tag{6}$$

so that the mth normal equation holds for polynomial

$$C_m(z) = C_{m-1}(z) + \kappa_m B_{m-1}(z) \tag{7}$$

which, by construction, already satisfies the first $m-1$ equations. Therefore, $C_m(z)$ solves the mth-order set of normal equations. Since $\mathbf{R}$ is a matrix with Toeplitz structure, it is possible to determine the corresponding backward prediction inverse filter by a simple reverse rearrangement of coefficients as

$$B_m(z) = z^{-(m+1)} C_m(1/z), \tag{8}$$

which preserves the polynomial norm so that its square norm may be recursively computed by

$$\beta_m = \left(1 - \kappa_m^2\right) \beta_{m-1} \tag{9}$$

due to the mutual orthogonality within the set of polynomials $\{B_l(z)\}_{l=0}^{m}$.

Finally, the first equation in set (1), which has been left out of consideration so far, must be solved. Let us say that polynomial $\overline{A}(z)$ satisfies it, that is,

$$\langle \overline{A}(z), 1\rangle = 1. \tag{10}$$

We know at this juncture that $\langle C_p(z), C_p(z)\rangle = \beta_p$ and, since $C_p(z)$ satisfies all equations in set (3), it results that

$$\langle C_p(z), 1\rangle = \beta_p \tag{11}$$

so that, by comparing Eq. (10) with Eq. (11), the complete solution to Eq. (1) is found to be vector $\mathbf{a}$ such that

$$a_i = \frac{c_{pi}}{\beta_p} \tag{12}$$

for $i = 0, 1, \ldots, p$, where $c_{p0} = 1$.

III. Levinson algorithm as an extension to Durbin's

For most work on speech analysis and coding, LP analysis involves the solution of a set of normal equations such as those in Eq. (1) [1]. However, for other applications in speech analysis such as discrete spectral modeling [5], [8] or in Levinson's original application to Wiener filtering [6], a full right-hand side vector emerges, leading to the set of equations

$$\mathbf{R}\mathbf{a} = \boldsymbol{h}, \tag{13}$$

where $\mathbf{R}$ is the $(p+1) \times (p+1)$ autocorrelation matrix and $h(i) \neq 0$ in general for $i = 0, 1, \ldots, p$. This set of equations may be described by means of polynomial inner products as

$$\langle A(z), z^{-j}\rangle = h(j), \quad j = 0, 1, \ldots, p, \tag{14}$$

where

$$A(z) = A_p(z) = a_{p0} + \sum_{i=1}^{p} a_{pi} z^{-i} \tag{15}$$

is the inverse LP filter that satisfies the set of equations (14). Note that now a_{p0} in Eq. (15) cannot be assumed to be unity the way c_{p0} was in Eq. (4). Therefore, in general, solution $A_p(z)$ will not belong to the same space as Durbin solution $C_p(z)$.

Fortunately, by-products of Durbin algorithm may be used as a shortcut to the solution. This can be readily understood by considering transformation

$$V(z) = zP(z), \tag{16}$$

which is isometric for any polynomial $P(z)$ under the inner product induced by $\mathbf{R}$ because

$$\langle zP(z), zQ(z)\rangle = \langle P(z), Q(z)\rangle \tag{17}$$

due to the Toeplitz nature of matrix $\mathbf{R}$. In particular, Eq. (17) holds in the following cases

$$\langle zB_m(z), zB_l(z)\rangle = \langle B_m(z), B_l(z)\rangle \tag{18}$$

for $m, l \in \{0, 1, \ldots, p\}$, providing enough evidence to the fact that $\{zB_m(z)\}_{m=0}^{p}$ is an orthogonal vector basis for the new polynomial space.

From Eq. (5) the backward prediction error filters for the second stage of Levinson algorithm are explicitly given by

$$zB_m(z) = z^{-m} + \sum_{i=0}^{m-1} b_{m,i+1} z^{-i} \tag{19}$$

for $m \in \{0, 1, \ldots, p\}$. Denote by $b_m(n)$ the mth-order first-stage Durbin backward prediction error signal, denote by $r_m(n)$ the mth-order second-stage Levinson backward prediction error signal, and let $s(n)$ stand for the original signal. Exciting $B(z)$ and $zB(z)$ with $s(n)$, according to Eqs. (5) and (19), they output the error signals

$$b_m(n) = s(n-m-1) + \sum_{i=1}^{m} b_{mi} s(n-i) \tag{20}$$

$$r_m(n) = s(n-m) + \sum_{i=0}^{m-1} b_{m,i+1} s(n-i). \tag{21}$$

The corresponding least square errors, by making $l = m$ in Eq. (18), are related by

$$\beta_m = \sum_{n=0}^{\infty} b_m^2(n) = \sum_{n=0}^{\infty} r_m^2(n). \tag{22}$$

With this support, the set of equations (14) may also be solved recursively in iterations of increasing LP order, starting by the first equation for the zeroth order as

$$\langle A_0(z), zB_0(z)\rangle = h(0). \tag{23}$$

Since a zeroth degree polynomial is a constant, Eq. (23) may be simplified to

$$A_0(z)\langle 1, 1\rangle = h(0) \tag{24}$$

or

$$A_0(z) = \frac{h(0)}{r_{00}}. \tag{25}$$

For higher prediction orders $m = 1, 2, \ldots, p$, the lower order inverse filter $A_{m-1}(z)$ and the next basis polynomial $zB_m(z)$

are linearly combined to construct the solution to the mth equation as

$$A_m(z) = A_{m-1}(z) + k_m z B_m(z), \tag{26}$$

where parameter k_m is assigned the value

$$k_m = \frac{h(m) - \langle A_{m-1}(z), z^{-m} \rangle}{\beta_m} \tag{27}$$

in just the right measure to make $A_m(z)$ the solution to the set of first $m+1$ equations in (14). When the order $m = p$ is reached, the solution $\mathbf{a}$ to Eq. (13) is found as

$$a_i = a_{pi} \tag{28}$$

for $i = 0, 1, \ldots, p$.

In conclusion, the principle of shifting the backward prediction from the past $(m+1)$th to the past mth sample, warranted by an extension of Durbin algorithm based on the Toeplitz property of the autocorrelation function, provides the basis to the derivation of the second stage of Levinson algorithm.

IV. Autocorrelation matrix inversion

Some applications require the iterative solution of sets of equations such as Eq. (13) with the same coefficient matrix as pointed out in Section I. In this situation, it is reasonable to consider inverting the coefficient matrix once and then use $\mathbf{R}^{-1}$ to reach the solution in each iteration ι by a matrix-vector product like

$$\mathbf{a}^{(\iota)} = \mathbf{R}^{-1}\mathbf{h}^{(\iota)}, \tag{29}$$

where $\mathbf{h}^{(\iota)}$ is the right-hand side vector for iteration ι.

Indeed, for symmetric Toeplitz matrices such as $\mathbf{R}$, the inverse matrix may be found in a simple way after having the solution $\mathbf{a}$ to the set of normal equations (1) as specified in Eq. (12). The inverse autocorrelation matrix may be obtained by one of GS relations [7] as

$$\mathbf{R}^{-1} = \frac{1}{\beta_p}\left(\mathbf{L}\left(\beta_p\mathbf{a}\right)\mathbf{L}^T\left(\beta_p\mathbf{a}\right) - \mathbf{L}(\mathbf{b})\mathbf{L}^T(\mathbf{b})\right), \tag{30}$$

where $\mathbf{L}(\mathbf{x})$ denotes the lower triangular Toeplitz matrix whose first column is vetor $\mathbf{x}$ and the components of vector $\mathbf{b}$ are defined by the coefficients of the pth-order backward prediction inverse filter $B_p(z)$ as

$$b_i = \begin{cases} 0, & i = 0 \\ b_{pi}, & 1 \le i \le p. \end{cases} \tag{31}$$

First, without ever evaluating $\mathbf{R}^{-1}$, let us replace it in Eq. (29) by the GS relation in Eq. (30) and split the computation into the following steps

$$\begin{aligned} \mathbf{r}^{(\iota)} &= \mathbf{L}^T(\mathbf{b})\mathbf{h}^{(\iota)} && (32)\\ \mathbf{s}^{(\iota)} &= \mathbf{L}^T\left(\beta_p\mathbf{a}\right)\mathbf{h}^{(\iota)} && (33)\\ \mathbf{t}^{(\iota)} &= \mathbf{L}(\mathbf{b})\mathbf{r}^{(\iota)} && (34)\\ \mathbf{u}^{(\iota)} &= \mathbf{L}\left(\beta_p\mathbf{a}\right)\mathbf{s}^{(\iota)}, && (35) \end{aligned}$$

where each matrix is triangular, resulting in the final calculation

$$\mathbf{a}^{(\iota)} = \frac{1}{\beta_p}\left(\mathbf{u}^{(\iota)} - \mathbf{t}^{(\iota)}\right) \tag{36}$$

for the computation of product (29) by means of GS decomposition (GSD).

In a different approach, the matrix factors may be operated on first while taking into consideration that matrix $\mathbf{R}$ is symmetric and so is its inverse. This leads to the determination of the elements in matrix $\mathbf{R}^{-1}$ by the following set of expressions

$$\begin{aligned} \left[\mathbf{R}^{-1}\right]_{00} &= a_0 \\ \left[\mathbf{R}^{-1}\right]_{0j} &= \beta_p a_j, \quad j = 1, 2, \ldots, p \\ \left[\mathbf{R}^{-1}\right]_{i0} &= \left[\mathbf{R}^{-1}\right]_{0i}, \quad i = 1, 2, \ldots, p \\ \left[\mathbf{R}^{-1}\right]_{ij} &= \beta_p \sum_{k=0}^{i} \left(a_{i-k}a_{j-k} - a_{p+1-i+k}a_{p+1-j+k}\right), \\ i &= 1, 2, \ldots, p, \quad j = i, i+1, \ldots, p \\ \left[\mathbf{R}^{-1}\right]_{ji} &= \left[\mathbf{R}^{-1}\right]_{ij}, \\ i &= 1, 2, \ldots, p-1, \quad j = i+1, i+2, \ldots, p \end{aligned} \tag{37}$$

Moreover, since $\mathbf{R}$ is a symmetric Toeplitz matrix, it is centrosymmetric, that is, the following relation holds

$$\mathbf{R} = \mathbf{JRJ} \tag{38}$$

where $\mathbf{J} = \begin{bmatrix} \delta_p & \delta_{p-1} & \ldots & \delta_0 \end{bmatrix}$ is the exchange matrix represented by its columns, which are in reverse order with respect to the corresponding identity matrix $\mathbf{I} = \begin{bmatrix} \delta_0 & \delta_1 & \ldots & \delta_p \end{bmatrix}$. The inverse autocorrelation matrix also inherits the centrosymmetric property as can be concluded from Eq. (38) since the exchange matrix is its own inverse. However, the Toeplitz property is lost by matrix inversion.

The centrosymmetric property allows the simplification of symmetric GS inversion from Eq. (37) to

$$\begin{aligned} \left[\mathbf{R}^{-1}\right]_{00} &= a_0 \\ \left[\mathbf{R}^{-1}\right]_{0j} &= \beta_p a_j, \quad j = 1, 2, \ldots, p \\ \left[\mathbf{R}^{-1}\right]_{i0} &= \left[\mathbf{R}^{-1}\right]_{0i}, \quad i = 1, 2, \ldots, p \\ \left[\mathbf{R}^{-1}\right]_{ip} &= \left[\mathbf{R}^{-1}\right]_{0,p-i}, \quad i = 1, 2, \ldots, p \\ \left[\mathbf{R}^{-1}\right]_{pj} &= \left[\mathbf{R}^{-1}\right]_{jp}, \quad j = 1, 2, \ldots, p \end{aligned}$$

for $i = 1, 2, \ldots, q$,

$$\begin{aligned} \left[\mathbf{R}^{-1}\right]_{ij} &= \left[\mathbf{R}^{-1}\right]_{i-1,j-1} + \beta_p\left(a_i a_j - a_{p+1-i}a_{p+1-j}\right), \\ &\qquad j = i, i+1, \ldots, p-i \\ \left[\mathbf{R}^{-1}\right]_{ji} &= \left[\mathbf{R}^{-1}\right]_{ij}, \qquad j = i+1, i+2, \ldots, p-i \\ \left[\mathbf{R}^{-1}\right]_{j,p-i} &= \left[\mathbf{R}^{-1}\right]_{i,p-j}, \quad j = i+1, i+2, \ldots, p-i \\ \left[\mathbf{R}^{-1}\right]_{p-i,j} &= \left[\mathbf{R}^{-1}\right]_{j,p-i}, \quad j = i+1, i+2, \ldots, p-i \end{aligned}$$

if p is even,

$$\begin{aligned} \left[\mathbf{R}^{-1}\right]_{q+1,q+1} = &\left[\mathbf{R}^{-1}\right]_{qq} \\ &+ \beta_p\left(a_{q+1}a_{q+1} - a_{p-q}a_{p-q}\right), \end{aligned} \tag{39}$$

where $q = \frac{p}{2} - 1$ when p is even and $q = \frac{p-1}{2}$ when p is odd.

V. Operational complexity

The task at hand is the solution of the full set of equations (13), which can be implemented as a single algorithm such as the Levinson transformation solution (LTS) detailed in

TABLE I
ARITHMETIC OPERATION COUNTS FOR LEVINSON TRANSFORMATION SOLUTION (LTS), GS DECOMPOSITION (GSD), CENTROSYMMETRIC GS (CSGS) MATRIX INVERSION AND MATRIX-VECTOR PRODUCT (MVP)

Algorithm	Operation count	Prediction order
LTS	$2p^2 + 3p$	p odd or even
GSD	$4p^2 + p + 1$	p odd or even
CSGS	$\frac{5}{4}p^2 - 5p$	p even
CSGS	$\frac{5}{4}p^2 - 5p + \frac{15}{4}$	p odd
MVP	$2p^2 + 2p$	p odd or even

TABLE II
ARITHMETIC OPERATION COUNTS FOR LEVINSON TRANSFORMATION SOLUTION (LTS), GS DECOMPOSITION (GSD) AND CENTROSYMMETRIC GS (CSGS) SOLUTION TO A FULL SET OF EQUATIONS AS FUNCTIONS OF PREDICTION ORDER AND NUMBER OF ITERATIONS (I)

Algorithm	Operation count	Prediction order
LTS	$(2p^2 + 3p)I$	p odd or even
GSD	$(4p^2 + p + 1)I$	p odd or even
CSGS	$\frac{5}{4}p^2 - 5p + (2p^2 + 2p)I$	p even
CSGS	$\frac{5}{4}p^2 - 5p + \frac{15}{4} + (2p^2 + 2p)I$	p odd

Section III. In applications such as discrete spectral modeling, this procedure has to be repeated for each iteration.

In a different arrangement, the autocorrelation matrix in Eq. (13) may be inverted once and then the solution is found by a matrix-vector product such as that in Eq. (29) in each iteration. An efficient method called centrosymmetric GS (CSGS) was presented in Section IV, where its operation is described by expressions (39).

The operational complexity of each algorithm involved in these approaches is presented in Table I as a function of the LP order p for a full set of $p+1$ equations. The whole operation counts for the solution of the set of equations are shown in Table II as functions of both the prediction order and the number of iterations.

The operation counts in Table II are exemplified for some particular cases of interest. A usual prediction order for discrete spectral modeling is $p = 14$ [9]. In this case, LTS is found to be better for up to 13 iterations. For higher prediction order such as 20 and 30, LTS is less complex for up to 19 and 32 iterations, respectively.

Continuous spectral modeling is usually performed at order $p = 10$ and occasionally also discrete spectral models of this order are fit [8], [9]. At this order CSGS operation count only becomes smaller after 7 iterations. Sometimes lower modeling orders are considered for a discrete spectral modeling stage as $p = 8$. In this case, LTS is less complex than CSGS up to the 4th iteration.

Since discrete spectral models usually converge in four or five iterations [5], LTS is seen to be advantageous for a wide range of prediction orders, possibly losing to CSGS at prediction orders lower than ten.

VI. CONCLUSION

The set of normal equations arising in LP is usually solved by Durbin algorithm, which is derived very elegantly in polynomial space. It is shown that a linear transformation of this polynomial space provides the setting for deriving the more general Levinson solution to a full set of equations. An efficient algorithm results because the transformation is isometric and builds upon the results of Durbin algorithm. Full sets of equations appear in discrete spectral modeling where they have to be solved in iterations for the same coefficient matrix. This makes it attractive to use an efficient Gohberg-Semencul relation for the inversion of the symmetric Toeplitz coefficient matrix. Further, an efficient implementation of this matrix inversion is proposed that uses to advantage its centrosymmetry. For a low number of iterations, arithmetic operation counts are found to be lower using the Levinson transformation algorithm, particularly for prediction orders above ten.

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