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Brief paper

Recursive linear estimation for general discrete-time descriptor systems

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ABSTRACT

This paper considers the optimal linear estimates recursion problem for discrete-time linear systems in its more general formulation. The system is allowed to be in descriptor form, rectangular, time-variant, and with the dynamical and measurement noises correlated. We propose a new expression for the filter recursive equations which presents an interesting simple and symmetric structure. Convergence of the associated Riccati recursion and stability properties of the steady-state filter are provided.

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1. Introduction

The optimal recursive estimation problem for discrete-time descriptor systems is dealt with in this paper. The study of estimation and control of descriptor systems (also known as singular systems or implicit systems) is motivated by the fact that systems in descriptor formulation frequently arise naturally in economical systems (Luenberger, 1977), image modeling (Hasan & Azimsadjadi, 1995), and robotics (Mills & Goldenberg, 1989). Recursive state estimations for discrete-time descriptor systems have been intensively studied in the literature (see for instance Dai, 1989; Darouach, Zasadzinski, & Mehdi, 1993; Deng & Liu, 1999; Germani, Manes, & Palumbo, 2001; Ishihara, Terra, & Campos, 2005; Nikoukhah, Campbell, & Delebecque, 1999; Nikoukhah, Willsky, & Levy, 1992; Zhang, Xie, & Soh, 1999). Different formulations have been proposed in order to deal with this problem. In particular, we have addressed the standard Kalman filter as a deterministic optimal trajectory fitting problem as reported in Ishihara et al. (2005).

In consequence, based on this approach we have solved the robust filtering problem through recursive algorithms, when there exist uncertainties in some matrices of the descriptor system (Ishihara, Terra, & Campos, 2006).

In this paper, the deterministic data fitting approach of Ishihara et al. (2005) is extended in order to consider the most general discrete-time linear system (in rectangular descriptor form, time-variant, and with correlation between the state and measurement noises allowed). In the Kalman filter literature, the direct treatment of this most general case is usually avoided through a previous change of variables. Even for simplified systems, where noise correlations do not exist, the final expressions of the filter equations are already rather cumbersome.

In this paper, we develop new expressions for the optimal recursive estimates of descriptor systems. These new expressions present interesting simple and symmetric structures. For instance, they open the possibility of generalizing the results obtained in Ishihara et al. (2006) in order to solve the most general robust filtering problem when uncertainties are present in all parameter matrices of the descriptor system.

The stability and convergence proofs of the proposed generalized Kalman filter use only deterministic arguments. This approach follows the philosophy adopted in Willems (2004) for continuous-time state–space systems. The descriptor Riccati equation analyzed in this paper generalizes the results presented in Lancaster and...
Rodman (1995) for state–space systems and can be considered as an alternative to the stochastic arguments used in Nikoukhah et al. (1992).

This paper is organized as follows. The deterministic estimation problem is formulated in Section 2. The predicted and filtered estimates are developed in Sections 3 and 4, respectively. Results that guarantee the stability and convergence of the Riccati recursion are presented in Sections 5 and 6 respectively. To conclude, a numerical example is given in Section 7.

2. Problem statement

Consider a set of measured signals \( z = \{ z_0, z_1, \ldots \} \) from a certain real dynamical system.

In its more general form, an ideal linear dynamical system which would 'explain' the measurements \( z \) up to the time instant \( k \) is a descriptor system described by

\[
E_{i+1}x_{i+1} = F_ix_i, \\
z_i = H_ix_i, \quad i = 0, 1, \ldots, k
\]

(1)

where \( x_i \) is the descriptor variable or state which describes the internal behavior of the system; \( E_i, F_i, \) and \( H_i \) are real rectangular matrices of appropriate dimensions. With the time instant \( k \) fixed, for each state sequence candidate \( \{ x_{0|k}, x_{1|k}, \ldots, x_{k|k} \} \) we can define implicitly the following fitting errors:

\[
\begin{bmatrix}
G_{w,i} \\
K_{w,i}
\end{bmatrix}
\begin{bmatrix}
w_{i,jk} \\
v_{i,kj}
\end{bmatrix} := \begin{bmatrix}
E_{i+1}x_{i+1} - F_ix_i \\
z_i - H_ix_i
\end{bmatrix}
\]

(2)

\[
\begin{bmatrix}
w_{i,jk} \\
v_{i,kj}
\end{bmatrix} := \begin{bmatrix}
E_{i+1}x_{i+1} - F_ix_i \\
z_i - H_ix_i
\end{bmatrix}
\]

for \( i = 0, 1, \ldots, k \). The matrices \( E_{i}, F_{i}, \) and the vector \( x_0 \) can deal with an a priori information on the initial state \( x_0 \), and are supposed of appropriate dimensions. Usually, it can be supposed that \( E_{0} = F_{1} = I \).

The deterministic optimal fitting approach adopted in this paper aims to find a state sequence which minimizes some predefined error functional. Once the minimizing sequence \( \{ \hat{x}_{i|k} \} \) has been obtained, we can define from (2) the corresponding minimum fitting errors \( \hat{w}_{i,jk}, \hat{v}_{i,kj}, \hat{p}_{0|0} \) so that the complete model which explains the set of measured signals up the instant \( k, z^k = \{ z_0, z_1, \ldots, z_k \} \), turns out to be

\[
\hat{F}_0\hat{x}_{0|0} = F_{0}\hat{x}_{0|0} + \hat{p}_{0|0}
\]

(3)

\[
z_0 = H_0\hat{x}_{0|0} + K_r\hat{v}_{0|0}
\]

(4)

\[
E_{i+1}\hat{x}_{i+1|j} = F_i\hat{x}_{i|j} + G_{w,i}\hat{w}_{i,jk} + G_{v,i}\hat{v}_{i,kj}
\]

(5)

\[
z_i = H_i\hat{x}_{i|j} + K_{w,i}\hat{w}_{i,jk} + K_{v,i}\hat{v}_{i,kj}, \quad i = 0, 1, \ldots, k.
\]

(6)

This model furnishes \( \hat{x}_{i+1|jk} \) and \( \hat{v}_{i|jk} \), i.e., the predicted and filtered estimates of the internal model (3)–(6) at the instant \( k \). The nomenclature 'filtering' is justified by the observation that if \( H_k = I \), \( K_{w,k} = 0 \) and \( K_{v,k} = I \) in the model (3)–(6), then we have from (6)

\[
z_i = \hat{x}_{i|jk} + \hat{v}_{i|jk},
\]

(7)

and so, if the signal \( \hat{v}_{i|jk} \) was obtained from the actually measured signal \( z_k \), the error signal \( \hat{v}_{i|jk} \) would be suppressed from \( z_k \). The optimality concept developed in this paper to solve the prediction problem is based on the functional cost (12) for \( k \geq 0 \). The positive definite weighting matrices proposed in this functional cost are related to the intuitive notion of (relative) degree of uncertainty, or how big we allow each fitting error to be.

Based on these considerations, the predicted data fitting problem is stated as (12)–(13) in the next section. For a simpler data fitting problem without crossed error weighting, it is shown in Ishihara et al. (2005) that the solution can be stated recursively in \( k \), leading to the standard Kalman filter for descriptor systems. Here, we consider Kalman filters for the most general case where the dynamical and measurement errors are cross-correlated, as shown in (12)–(13). We suppose that the model parameters \( E_i, F_i, G_{w,i}, \) \( G_{v,i} \), \( H_i, K_{w,i}, \) and \( K_{v,i} \) are known. The model is considered consistent; that is, (2) has at least one solution for every \( k \). We also suppose known the positive definite weighting matrices \( Q_i, R_i, P_0 \) for the errors \( w_{i|jk}, v_{i|jk} \), and \( p_{0|0} \), respectively, and the cross–term weighting matrix \( S_j \).

Remark 1. It is worth observing that the aforementioned deterministic filtering problem also solves the equivalent stochastic filtering problem, which is defined based on the following discrete-time linear descriptor system:

\[
E_{i+1}x_{i+1} = F_ix_i + G_{w,i}w_i + G_{v,i}v_i,
\]

\[
z_i = H_ix_i + K_{w,i}w_i + K_{v,i}v_i
\]

(8)

where \( i = 0, 1, 2, \ldots \), \( x_i \in \mathbb{R}^n \) is the descriptor variable, \( z_i \in \mathbb{R}^m \) is the measured output, and \( w_i \in \mathbb{R}^m \) and \( v_i \in \mathbb{R}^p \) are the state and the measurement noises. The initial condition \( x_0 \in \mathbb{R}^n \) is a random variable such that \( E_0x_0 \) has mean value \( F_{i+1}\hat{x}_{0|k} \) and covariance \( P_0 \); \( w_i \) and \( v_i \) are zero-mean, independent of \( x_0 \), white sequences with known covariance matrices:

\[
E\left[ \begin{bmatrix}
w_i \\
v_i
\end{bmatrix} \begin{bmatrix}
w_j \\
v_j
\end{bmatrix}^T \right] = \begin{bmatrix}
Q_j & S_j \\
S_j^T & R_j
\end{bmatrix} \delta_{ij}
\]

(9)

where \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) otherwise. In this context, the Kalman filters can be developed to estimate recursively the following problems:

(i) the linear least–mean–squares predicted estimate

\[
\hat{x}_{i+1|jk} = E[ x_{i+1|jk} | z_k, z_{k-1}, \ldots, z_0 ]
\]

(10)

(ii) the linear least–mean–squares filtered estimate

\[
\hat{x}_{i|jk} = E[ x_{i|jk} | z_k, \ldots, z_0 ]
\]

(11)

The purpose of this paper is to present, for linear systems in their most general case, a very simple expression for the recursions of \( \hat{x}_{i+1|jk} \) and \( \hat{x}_{i|jk} \). The predicted recursion is derived from solution of the problem (12)–(13). The filtered data fitting problem can be stated similarly to the predicted problem.

3. Singular Kalman predictor

The deterministic predicted least-square data fitting problem is to find a sequence \( \{ \hat{x}_{0|0}, \hat{x}_{1|1}, \ldots, \hat{x}_{k+1|k} \} \) which minimizes the following fitting error cost:

\[
\mathcal{J}_k(\{ x_{i|jk} \}_{i=0}^{k+1} | k+1 ) := \frac{1}{2} \left[ \| F_0x_0 - F_{i+1}\hat{x}_{0|j} \|_R^{-1} \right]^2
\]

(12)

\[
\quad + \sum_{i=0}^{k} \left[ w_{i,jk} \right]^T \begin{bmatrix}
Q_j & S_j \\
S_j^T & R_j
\end{bmatrix}^{-1} \left[ w_{i,jk} \right]
\]

subject to

\[
E_{i+1}x_{i+1} = F_ix_i + G_{w,i}w_i + G_{v,i}v_i,
\]

\[
z_i = H_ix_i + K_{w,i}w_i + K_{v,i}v_i, \quad k \geq 0.
\]

(13)

To simplify the notation, we introduce the following auxiliary variables:

\[
F_i := \begin{bmatrix} F_i \\ H_i \end{bmatrix}, \quad Z_i := \begin{bmatrix} 0 \n 0 \n z_i \end{bmatrix},
\]

\[
R_i := \begin{bmatrix} Q_j & S_j \\
S_j^T & R_j \end{bmatrix}, \quad \hat{F}_i := \begin{bmatrix} G_{w,i} & G_{v,i} \\
K_{w,i} & K_{v,i} \end{bmatrix},
\]

for \( 0 \leq i \leq k \) and \( \delta_i := \begin{bmatrix} 0 \\ 0 \end{bmatrix} \).
for 1 ≤ j ≤ k + 1,

\[ R_k := \begin{bmatrix} R_{k-1} & 0 \\ 0 & R_0 \end{bmatrix}, \quad Z_k := \begin{bmatrix} -F_{k-1} \tilde{x}_0 \\ z_0 \end{bmatrix}, \quad \eta_0 := \begin{bmatrix} I & 0 \\ 0 & K_{v,0} \end{bmatrix}, \quad w_{-1|0} := E_0 z_{0|0} - F_{-1} \tilde{x}_0. \]  

(14)

In the expressions above, we adopt \( Q_{-1} := P_0, S_{-1} = S^T_{-1} = 0, \) \( G_{w,-1} := I, G_{v,-1} := 0, K_{v,-1} := 0. \)

The solution of (12)-(13) follows the arguments developed in Ishihara et al. (2005). The only difference is that Lagrange multipliers are used to incorporate the constraints (13) that do not exist in Ishihara et al. (2005). The main result of this section is presented in the following.

**Theorem 3.1.** Suppose that the assumptions

A 1. \( \begin{bmatrix} E_{k+1} & F_{k} & G_{w,k} & G_{v,k} & 0 \\ 0 & H_{k} & K_{w,k} & K_{v,k} & 0 \end{bmatrix} \) has full row rank;

A 2. \( E_{i+1} \) has full column rank;

hold for all 0 < i ≤ k and there is given a sequence \( \{z_0, z_1, \ldots, z_k\}. \)

Then the successive optimal estimates \( \tilde{x}_{k+1|k}, w_{k+1|k} \) leading to the solution of the data fitting problem (12)-(13) can be obtained from the following recursion:

\[
\begin{bmatrix} \tilde{x}_{0|0} & P_{0|0} \\ \tilde{x}_{k+1|k} & P_{k+1|k} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{x}_{k|k-1} & 0 \\ 0 & 0 \end{bmatrix}.
\]

(15)

In this theorem, assumption A 1 is related to the consistency of the system parameters and assumes that the system does not furnish redundant information. Note that in Nikoukhah et al. (1992) it was shown that a similar condition ensures the use of inverses of a block matrix instead of pseudo-inverses. Assumption A 2 is related to the estimability and it ensures the existence of a unique optimal filter (Nikoukhah et al., 1992). Observe that although we have considered a more general linear system than the usual models considered in the literature, the final expressions of (15) are very simple and have a central matrix with symmetric structure.

In order to compare Theorem 3.1 with the results reported in Ishihara et al. (2005), we consider \( G_{v,k} = 0, K_{w,k} = 0, K_{v,k} = I, \) \( S_k = 0. \) If in addition \( F_k, G_{w,k} \) has full row rank, we can rewrite (15) as

\[
\begin{bmatrix} \tilde{x}_{k+1|k} & P_{k+1|k} \end{bmatrix} := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{x}_{k|k-1} & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} Y_k & -F_k P_{k|k-1} H_k^T & -F_k P_{k|k-1} H_k^T E_{k+1} & 0 \\ I & 0 & 0 & 0 \end{bmatrix}^{-1} \times \begin{bmatrix} F_k P_{k|k-1} H_k & H_k P_{k|k-1} F_k^T & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \tilde{x}_{k+1|k} & -F_k P_{k|k-1} H_k & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

(16)

or, by applying the matrix inversion lemma, as

\[
\begin{bmatrix} \tilde{x}_{k+1|k} & P_{k+1|k} \end{bmatrix} := \begin{bmatrix} F_k P_{k|k-1} F_{k+1}^T \end{bmatrix}^{-1} \begin{bmatrix} Y_k & -F_k P_{k|k-1} H_k^T & -F_k P_{k|k-1} H_k^T E_{k+1} & 0 \\ I & 0 & 0 & 0 \end{bmatrix}^{-1} \times \begin{bmatrix} F_k P_{k|k-1} H_k & H_k P_{k|k-1} F_k^T & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \tilde{x}_{k|k-1} & -F_k P_{k|k-1} H_k & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

(17)

4. **Singular Kalman filter**

The deterministic filtered least-square data fitting problem aims to minimize the following fitting error cost. Considering the filtered case, we have that the minimization problem (12)-(13) can be modified as

\[
\begin{bmatrix} E_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_0 z_{0|0} - F_{-1} \tilde{x}_0 \end{bmatrix}^2 + \begin{bmatrix} u_{i|k} \end{bmatrix}^2 \begin{bmatrix} 0 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}
\]

subject to

\[
E_{i+1} X_{i+1} = E_i X_i + \sum_{l=0}^{k-1} w_{l|k+1} \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}
\]

(18)

The auxiliary variables (14) are redefined as

\[
E_j := \begin{bmatrix} -E_j \\ H_j \end{bmatrix}, \quad F_{j-1} := \begin{bmatrix} F_{j-1} \\ J_{j-1} \end{bmatrix},
\]

for 0 ≤ j ≤ k,

\[
E_i := \begin{bmatrix} Q_{i-1} & S_{i-1} & 0 & 0 \\ S_{i-1}^T & R_i \end{bmatrix}, \quad \eta_i := \begin{bmatrix} G_{w,i-1} & G_{v,i} \\ K_{w,i-1} & K_{v,i} \end{bmatrix}, \quad Z_i := \begin{bmatrix} 0 & 0 \\ Z_i & 0 \end{bmatrix},
\]

for 1 ≤ i ≤ k. The optimal recursive estimates and the corresponding Riccati recursion are given by

\[
\begin{bmatrix} \tilde{x}_{k|k} & P_{k|k} \end{bmatrix} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_{k+1} & 0 \\ 0 & E_{k+1} \end{bmatrix}^{-1} \begin{bmatrix} Y_k & -F_k P_{k|k-1} H_k^T & -F_k P_{k|k-1} H_k^T E_{k+1} & 0 \\ I & 0 & 0 & 0 \end{bmatrix}^{-1} \times \begin{bmatrix} F_k P_{k|k-1} H_k & H_k P_{k|k-1} F_k^T & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \tilde{x}_{k+1|k} & 0 \\ 0 & Z_k \end{bmatrix}.
\]

(19)

Observe that in (18) the measured signal can be depending on the current and on delayed states, where \( J_i \) is the matrix of the delayed term (see Bianco, Ishihara, & Terra, 2005 for more details). Notice that a “3-block” form of the filtered case can be obtained.
directly from (19), or alternatively, following the approach developed in Nikoukhah et al. (1999), as
\[
\hat{x}_{k|k} = -\begin{bmatrix} 0 & 0 & I \\
\hat{S}_k & \hat{G}_k & -\hat{E}_k \\
\hat{E}_k & H_k & 0 \\
\end{bmatrix}^{-1} \begin{bmatrix} \hat{F}_{k-1} \hat{x}_{k-1} \\
\hat{S}_k \hat{x}_{k-1} \\
\hat{E}_k \\
\end{bmatrix} + \begin{bmatrix} 0 & 0 & I \\
\hat{G}_k & -\hat{E}_k & H_k \\
\hat{E}_k & 0 & 0 \\
\end{bmatrix} \begin{bmatrix} G_{w,k-1} \hat{Q}_{k-1} \hat{G}_{w,k-1} \\
G_{v,k-1} \hat{Q}_{k-1} \hat{G}_{v,k-1} \\
G_{v,k} \hat{K}_{v,k} \\
\end{bmatrix}
\]
(20)
where
\[
\hat{F}_{k-1} := F_{k-1} P_{k-1} \hat{F}_{k-1} + G_{w,k-1} \hat{Q}_{k-1} \hat{G}_{w,k-1} + G_{v,k} \hat{R}_k \hat{G}_{v,k}; \\
\hat{S}_k := F_{k-1} P_{k-1} \hat{S}_{k-1} \hat{S}_{k-1} + G_{w,k-1} \hat{Q}_{k-1} \hat{G}_{w,k-1} + G_{v,k} \hat{R}_k \hat{G}_{v,k}; \\
\hat{G}_k := F_{k-1} P_{k-1} \hat{G}_{k-1} + G_{w,k-1} \hat{Q}_{k-1} \hat{G}_{w,k-1} + G_{v,k} \hat{R}_k \hat{G}_{v,k}; \\
\]
If we consider the parameter matrices to be time-invariant, \(S_{k-1} = G_{v,k-1} = 0\) and \(J_k = 0\), this filter collapses to the filter developed in Nikoukhah et al. (1999). It is expected that, for systems with sparse matrices, the computation of (19) may be numerically more interesting than the computation of (20). This is an advantage of the proposed formulation if compared to the filters developed in Nikoukhah et al. (1999), and Zhang et al. (1999).

5. Stability of the steady-state estimator

In order to verify the stability of the steady-state estimator, we consider that the parameters in (14) are time-invariant and we introduce the following notation:
\[
\Omega (P) := \begin{bmatrix} P & 0 & 0 & 0 & 0 & 0 \\
0 & \mathcal{R} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathcal{F} & \mathcal{g} & \mathcal{e} \\
1 & 0 & \mathcal{g}^T & 0 & 0 & 0 \\
0 & 1 & \mathcal{g}^T & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \geq 0
\]
(21)
ei := [0 \cdots 1 \cdots 0]^T, \quad M (P) := \Omega^{-1} (P)
where \(e_i\) and \(M (P)\) are partitioned in blocks according to the block partition of \(\Omega (P)\). \(M (P)\) is partitioned as \(M_{ij}, i, j = 1, \ldots, 6\) and the vector of blocks \(e_i\) has the identity matrix at the \(i\)-th block position and zero matrices at the block positions at zero order. Also, we write simply \(\hat{x}_k\) and \(P_k\) instead of the corresponding \(\hat{x}_{k+1|k}\) and \(P_{k+1|k}\) and the steady-state value of \(P_{k+1}\) is denoted as \(P\).

The matrices \(\mathcal{R}, \mathcal{F}, \mathcal{g}, \mathcal{e}\) are time-invariant counterparts of the corresponding variables defined in (14). With the introduced notation we can show that the steady-state filter is given by
\[
\hat{x}_k = M_0 \hat{x}_{k-1} + M_{63} Z_k
\]
(22)
\[
P = -M_{66}.
\]
(23)
The steady-state filter (22)–(23) can be rewritten as
\[
\hat{x}_k = -M_{63} \mathcal{R} \hat{x}_{k-1} + M_{63} Z_k \quad \text{and} \quad P = M_{60} \mathcal{R} M_{61}^T + M_{62} \mathcal{R} M_{61}^T.
\]
(24)
(25)
Definition 2. \(P\) is a stabilizing solution of the algebraic Riccati equation (ARE) (23) if \(P\) satisfies (23) and \(M_{61}\) is stable (the matrix \(M_{61}\) has all eigenvalues inside the unit circle \(|\lambda| < 1\).

The main result of this section will be presented in the following. We will show that the ARE has a stabilizing semidefinite solution \(P\).

Theorem 5.1. Suppose that \(\mathcal{R} \geq 1, \mathcal{R} > 0\) and \(\mathcal{F}\) has full column rank. Let \(P\) be a solution of the ARE (23). If \(P \geq 0\) then \(P\) is the unique stabilizing solution for (23).

Proof. In Bianco et al. (2005) we proved that \(P \geq 0\) is stabilizing. It remains to prove that the stabilizing solution is unique. Unicity: Suppose that there exist two stabilizing solutions \(P_1\) and \(P_2\) for (23) and define the corresponding functions \(\Omega_1 = \Omega (P_1), \Omega_2 = \Omega (P_2), M_1 = M (P_1)\) and \(M_2 = M (P_2)\). Thus,
\[
p_1 - p_2 = M_{66}^2 - M_{66}^1 = e_1^T \Omega_1^{-1} (\Omega_1 - \Omega_2) e_1.
\]
But \(\Omega_1 - \Omega_2 = e_1 (P_1 - P_2) e_1\); then
\[
p_1 - p_2 = M_{61} (P_1 - P_2) M_{61}^T.
\]
(26)
As \(M_{61}\) and \(M_{62}^2\) are stable, the Stein equation (26) has a unique solution \(P_1 - P_2 = 0\). \(\diamond\)

6. Convergence

In this section we present some results that guarantee convergence of the generalized Riccati recursion, i.e., we show that the Riccati recursion is a monotone nondecreasing sequence, bounded by \(P \geq 0\). In Nikoukhah et al. (1992), the convergence of the Riccati recursion is proved through stochastic arguments. In this paper we use deterministic approaches, which have been useful in solving robust filtering problems (Ishihara et al., 2006). Notice that we are showing only the convergence proof for the predicted case. For the filtered case the convergence proof can be similarly obtained.

Lemma 3. Consider a sequence \(\{P_k\}\) generated by the recursion (15). If \(P_{k-1} \geq 0\), then \(P_k\) is positive semidefinite.

Proof. The Riccati recursion (15) can be rewritten as
\[
P_k = M_{61,k-1} P_{k-1} M_{61,k-1}^T + M_{62,k-1} M_{62,k-1}^T.
\]
Since \(\mathcal{R} > 0\) and by hypothesis \(P_{k-1} \geq 0\), it follows that \(P_k \geq 0\). \(\diamond\)

Lemma 4. Consider the following matrices, for \(i = 1, 2:\)
\[
P_k = M_{66,k-1}^i \quad \text{and} \quad F_k^i := M_{61,k-1}^i.
\]
(27)
Then, we obtain
\[
(i) \quad P_k^i - P_k^2 = F_k^i (P_k^1 - P_k^{1,i-1}) (F_k^2)^T
\]
(28)
(ii) \(F_k^i - F_k^2 = F_k^i (P_k^1 - P_k^{1,i-1}) M_{1,1,k-1}^i.
\]
(29)
Proof. First, we prove the relation (i). Subtracting \(P_k^{2,i} = P_k^{1,i+1}\) from \(P_k^i\), we obtain
\[
P_k^i - P_k^{2,i} = M_{66,k}^i - M_{66,k}.
\]
Defining \(\Omega_k^i := (P_k^i)^{-1}\) for \(i = 1, 2\), it follows that
\[
P_k^i + P_k^{2,i} + e_1^i (\Omega_k^i)^{-1} (\Omega_k^1 - \Omega_k^2) (\Omega_k^2)^{-1} e_1^i.
\]
As \(\Omega_k^1 - \Omega_k^2 = e_1 (P_k^1 - P_k^2) e_1^T\), we obtain
\[
P_k^i + P_k^{2,i} + e_1^i (\Omega_k^1)^{-1} (P_k^1 - P_k^2) e_1^T (\Omega_k^2)^{-1} e_1 = e_1 (P_k^1 - P_k^2) e_1^T (\Omega_k^2)^{-1} e_1.
\]
As \(\Omega_k^2\) is a symmetric matrix,
\[
P_k^i + P_k^{2,i} = (M_{61,k}) (P_k^1 - P_k^2) (M_{61,k}^2)^T.
\]
Thus, we obtain (i). The proof of part (ii) is analogous. \(\diamond\)
Lemma 5. Consider (27) for $i = 1, 2$. If $p_{k-1}^1 \geq p_{k-1}^2 \geq 0$, then $p_k^1 \geq p_k^2 \geq 0$.

Proof. By Lemma 4, we have relation (28). Furthermore,
\[
F_k^1 = F_k^2 (I + (p_{k-1}^1 - p_{k-1}^2) M_{11,k-1}^1).
\]
Thus,
\[
p_k^1 - p_k^2 = F_k^1 (I + (p_{k-1}^1 - p_{k-1}^2) M_{11,k-1}^1 (p_{k-1}^1 - p_{k-1}^2)) (F_k^2)^T.
\]
Since $p_{k-1}^1 - p_{k-1}^2 \geq 0$,
\[
p_k^1 - p_k^2 = F_k^2 (p_{k-1}^1 - p_{k-1}^2)^2 (I + (p_{k-1}^1 - p_{k-1}^2)^2)
\]
\[
\times M_{11,k-1}^1 (p_{k-1}^1 - p_{k-1}^2) \geq 0.
\]
Moreover, $M_{11,k-1}^1$ is given by
\[
M_{41,k-1} + M_{42,k-1} R M_{42,k-1}^T - M_{41,k-1} R M_{42,k-1}^T
\]
As $p_{k-1}^1$ and $R$ are semidefinite positive, it follows that $M_{11,k-1}^1 \geq 0$. Then, by Eq. (30), we obtain $p_k^1 - p_k^2 \geq 0$. 

Lemma 6. Suppose that $R > 0$ and define $\{p_k\}_{k=0}^\infty$ by
\[
p_k := M_{40,k-1} (F P_k F^T + g R g^T) M_{41,k-1}^{-1}
\]
with $p_0 = 0$. Then $\{p_k\}_{k=0}^\infty$ is a nondecreasing monotone sequence.

Proof. Since $R > 0$ and $p_k$ is given by (31), we have that for $p_0 = 0$ it follows that $p_1 \geq 0$ and $p_1 - p_0 \geq 0$. Consider, by induction, that the hypothesis $p_k - p_{k-1} \geq 0$ yields. By Lemma 5, if $p_k \geq p_{k-1}$, then $p_k \leq p_{k+1}$. Thus, it follows that $0 = p_0 \leq p_1 \leq \cdots \leq p_{k-1} \leq p_k \leq p_{k+1} \leq \cdots$.

Lemma 7. The Riccati recursion given by (15) can be rewritten in the following form:
\[
P_k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & E^T \\ 0 & E^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} E \end{bmatrix}
\]
where we define $E_k := [I \ -1_{p,k}]$ and
\[
P_k := L_k \begin{bmatrix} F & g \end{bmatrix} \begin{bmatrix} p_{k-1} & 0 \\ 0 & R \end{bmatrix}^{1/2} L_k^T.
\]

Lemma 8. Let $E$ be full column rank and $R > 0$. Consider a given arbitrary sequence $\{M_k\}_{k=0}^\infty$ and a matrix $Y_0 \geq 0$, and let the sequence $\{Y_k\}_{k=0}^\infty$ be defined by (33). Let $P_0$ be a matrix for which $0 \leq p_0 \leq Y_0$ and define a sequence $\{P_k\}_{k=0}^\infty$ (32). Then $0 \leq p_k \leq Y_k$ for $k = 0, 1, 2, \ldots$

Lemma 9. Let the sequence $\{p_k\}_{k=0}^\infty$ be defined as in Lemma 7. If $\lambda E + F$ has full column rank for $|\lambda| \geq 1$, then there exists a matrix $P^*$ such that $0 \leq p_k \leq P^*$ for $k = 0, 1, 2, \ldots$

The proofs of Lemmas 8 and 9 are extensions of Lancaster and Rodman (1995), which in turn is due to Caines and Mayne (1970).

Theorem 6.1. Assume that $\lambda E + F$ has full column rank for $|\lambda| \geq 1$ and $R > 0$. Let a sequence $\{P_k\}_{k=0}^\infty$ be generated by (15), with $P_0 = 0$. Then, it is a nondecreasing sequence and converges to $P^* \geq 0$, which satisfies the Riccati equation:
\[
P = M_{40,k} (F P F^T + g R g^T) M_{41,k}^{-1}.
\]

Proof. Starting with $P_0 = W_0 = 0$, generate two sequences of matrices $\{P_k\}$ and $\{W_k\}$ by (34) and
\[
W_k := M_{40,k} (F W_{k-1} F^T + g R g^T) M_{41,k}^{-1}.
\]
Then, Lemma 8 implies that $p_k \leq W_{k-1}$ and
\[
p_k - W_{k-1} = M_{40,k} (p_{k-1} - W_{k-2}) M_{41,k}^{-1}.
\]
Now $P_0 = 0$ implies that $p_1 \geq 0$, so $p_1 \leq W_1 \geq 0$. By the induction hypothesis $p_k - W_{k-2} \geq 0$ and by (36), $p_k \geq W_{k-1}$. Consequently, $p_k - W_{k-1} \leq p_k$ and, by Lemma 9, $\{p_k\}_{k=0}^\infty$ is a nondecreasing sequence with an upper bound.

Remark 10. Regarding the global stability of the filters proposed in this paper, we can see for example that the dynamic of the unforced part of the filter (23) is given by the matrix $M_{40,k}$. In this way it is expected that this class of stability could be proved by inclusion of appropriate notions of controllability and observability for time-varying systems. In general these concepts have been developed in the state–space literature based on stochastic arguments; see for instance (Kamen & Su 1999) and references therein. To follow this line, it is necessary to suppose the invertibility of the parameter matrix $F_k$ of the system. This is a very restrictive assumption in the descriptor context. As future work we intend to develop an appropriate procedure to check the global stability of this filter through a deterministic approach.

7. Numerical example

This example considers the descriptor system, with correlated noises, described by (13), where the system matrices are given by
\[
E_{k+1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad F_k = \begin{bmatrix} 0.3 & 0 \\ 0.2 & 0 \end{bmatrix}; \quad G_{v,k} = \begin{bmatrix} 0.34 & 0.21 \\ 1 & 2.7 \end{bmatrix};
\]
\[
G_{w,k} = \begin{bmatrix} 0.1 & 0.4 \\ 0.6 & 0.8 \end{bmatrix}; \quad H_k = \begin{bmatrix} 1.4 & 0.8 \end{bmatrix};
\]
\[
K_{v,k} = \begin{bmatrix} 1.4 & 1.4 \end{bmatrix}; \quad K_{v,k} = 1;
\]
and the covariances of $w_k$ and $v_k$ (with the cross-terms) are given respectively by
\[
Q_k = \begin{bmatrix} 0.0002 & 0.0002 \\ 0.0002 & 0.001 \end{bmatrix}; \quad R_k = 0.5; \quad S_k = \begin{bmatrix} 0.01 & 0.0001 \end{bmatrix}.
\]
The simulation results based on the filter developed in Theorem 3.1 are presented in Fig. 1.
8. Conclusion

We have considered the Kalman estimation problem for singular linear systems in its more general formulation where all system parameters are considered at once. We have introduced a “9-block” form for the filter and Riccati equation which present an interesting simple and symmetric structure. In this case, we have observed that it is natural, in this framework, to solve the filtering problem with a one-step delayed state. We are interested in this form mainly to analyze the effect of uncertainties on all system matrices in order to generalize the results of Ishihara et al. (2006).

References


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