THE LOWER CENTRAL AND DERIVED SERIES OF THE BRAID GROUPS OF THE SPHERE

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THE LOWER CENTRAL AND DERIVED SERIES OF THE BRAID GROUPS OF THE SPHERE

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ABSTRACT. In this paper, we determine the lower central and derived series for the braid groups of the sphere. We are motivated in part by the study of Fadell-Neuwirth short exact sequences, but the problem is important in its own right.

The braid groups of the 2-sphere $S^2$ were studied by Fadell, Van Buskirk and Gillette during the 1960s, and are of particular interest due to the fact that they have torsion elements (which were characterised by Murasugi). We first prove that for all $n \in \mathbb{N}$, the lower central series of the $n$-string braid group $B_n(S^2)$ is constant from the commutator subgroup onwards. We obtain a presentation of $\Gamma_2(B_n(S^2))$, from which we observe that $\Gamma_2(B_4(S^2))$ is a semi-direct product of the quaternion group $Q_8$ of order 8 by a free group $F_2$ of rank 2. As for the derived series of $B_n(S^2)$, we show that for all $n \geq 5$, it is constant from the derived subgroup onwards. The group $B_n(S^2)$ being finite and soluble for $n \leq 3$, the critical case is $n = 4$ for which the derived subgroup is the above semi-direct product $Q_8 \rtimes F_2$. By proving a general result concerning the structure of the derived subgroup of a semi-direct product, we are able to determine completely the derived series of $B_4(S^2)$ which from $(B_4(S^2))^{(4)}$ onwards coincides with that of the free group of rank 2, as well as its successive derived series quotients.

1. Introduction

1.1. Generalities and definitions. Let $n \in \mathbb{N}$. The braid groups of the plane $E^2$, denoted by $B_n$ and known as Artin braid groups, were introduced by E. Artin in 1925 [1, 2, 3] and admit the following well-known presentation: $B_n$ is generated by elements $\sigma_1, \ldots, \sigma_{n-1}$, subject to the classical Artin relations:

\[
\begin{align*}
\sigma_i \sigma_j & = \sigma_j \sigma_i \text{ if } |i - j| \geq 2 \text{ and } 1 \leq i, j \leq n - 1, \\
\sigma_i \sigma_{i+1} \sigma_i & = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for all } 1 \leq i \leq n - 2.
\end{align*}
\]

A natural generalisation to braid groups of arbitrary topological spaces was made at the beginning of the 1960s by Fox (using the notion of configuration space) [8]. The braid groups of compact, connected surfaces have been widely studied, and (finite) presentations were obtained in [3, 4, 11, 15, 16]. As well as being interesting in their own right, braid groups have played an important role in many branches of mathematics; for example in topology, geometry, algebra and dynamical systems,
and notably in the study of knots and links [9], of mapping class groups [6] [7], and of configuration spaces [10] [12]. The reader may consult [6] [34] [40] for some general references on the theory of braid groups.

Let $M$ be a connected manifold of dimension 2 (or surface), perhaps with boundary. Further, we shall suppose that $M$ is homeomorphic to a compact 2-manifold with a finite (possibly zero) number of points removed from its interior. We re-
nary. Further, we shall suppose that $M$ is a topological disc, there is a group homomorphism $\nu: B_n(\mathbb{D}^2) \to B_n(\mathbb{S}^2)$ induced by the inclusion. If $\beta \in B_n(\mathbb{D}^2)$, then its image $i(\beta)$ shall be denoted simply by $\beta$. It is well known that $B_n(\mathbb{S}^2)$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$ which are subject to the relations (1.1), plus the following relation, known as the surface relation of $B_n(\mathbb{S}^2)$:

$$\sigma_1 \cdots \sigma_{n-2}^2 \sigma_{n-1} \cdots \sigma_1 = 1.$$ 

Hence $B_n(\mathbb{S}^2)$ is a quotient of $B_n$. The first three sphere braid groups are finite: $B_1(\mathbb{S}^2)$ is trivial, $B_2(\mathbb{S}^2)$ is cyclic of order 2, and $B_3(\mathbb{S}^2)$ is a $\mathbb{Z}$-$S$-metacyclic group

Let $F_n(M) = \{(x_1, \ldots, x_n) \mid x_i \in M \text{ and } x_i \neq x_j \text{ if } i \neq j\}$. Since $F_n(M)$ is a subspace of the $n$-fold Cartesian product of $M$ with itself, the topology on $M$ induces a topology on $F_n(M)$. Then we define the $n$-string pure braid group $P_n(M)$ of $M$ to be $P_n(M) = \pi_1(F_n(M))$. There is a natural action of the symmetric group $S_n$ on $F_n(M)$ by permutation of coordinates, and the resulting orbit space $F_n(M)/S_n$ shall be denoted by $D_n(M)$. The fundamental group $\pi_1(D_n(M))$ is called the $n$-string (full) braid group of $M$, and shall be denoted by $B_n(M)$. Notice that the projection $F_n(M) \to D_n(M)$ is a regular $n!$-fold covering map.

The second definition of surface braid groups is geometric. Let $\mathcal{P} = \{p_1, \ldots, p_n\}$ be a set of $n$ distinct points of $M$. A geometric braid of $M$ with basepoint $\mathcal{P}$ is a collection $\beta = (\beta_1, \ldots, \beta_n)$ of $n$ paths $\beta: [0,1] \to M$ such that:

1. for all $i = 1, \ldots, n$, $\beta_i(0) = p_i$ and $\beta_i(1) \in \mathcal{P}$.
2. for all $i, j = 1, \ldots, n$ and $i \neq j$, and for all $t \in [0,1]$, $\beta_i(t) \neq \beta_j(t)$.

Two geometric braids are said to be equivalent if there exists a homotopy between them through geometric braids. The usual concatenation of paths induces a group operation on the set of equivalence classes of geometric braids. This group is isomorphic to $B_n(M)$ and does not depend on the choice of $\mathcal{P}$. The subgroup of pure braids, satisfying additionally $\beta_i(1) = p_i$ for all $i = 1, \ldots, n$, is isomorphic to $P_n(M)$. There is a natural surjective homomorphism $\tau: B_n(M) \to S_n$ which to a geometric braid $\beta$ associates the permutation $\tau(\beta)$ defined by $\beta_i(1) = p_{\tau(\beta)(i)}$. The kernel is precisely $P_n(M)$, and we thus obtain the following short exact sequence:

$$1 \to P_n(M) \to B_n(M) \xrightarrow{\tau} S_n \to 1.$$ 

It is well known that $B_n$ (resp. $P_n$) is isomorphic to $B_n(\mathbb{D}^2)$ (resp. $P_n(\mathbb{D}^2)$), where $\mathbb{D}^2$ is the closed 2-disc.

In this paper, we shall be primarily interested in the braid groups of the 2-sphere $\mathbb{S}^2$. Along with the braid groups of the real projective plane, they are of particular interest, notably because they have non-trivial centre (which is also the case for the Artin braid groups) and torsion elements (which were characterised by Murasugi [28]). We briefly recall some of their properties [11] [14] [19] [44]. If $\mathbb{D}^2 \subseteq \mathbb{S}^2$ is a topological disc, there is a group homomorphism $\nu: B_n(\mathbb{D}^2) \to B_n(\mathbb{S}^2)$ induced by the inclusion. If $\beta \in B_n(\mathbb{D}^2)$, then its image $i(\beta)$ shall be denoted simply by $\beta$.
(a group whose Sylow subgroups, commutator subgroup and commutator quotient group are all cyclic) of order 12. If \( n \geq 4 \) then \( B_n(S^2) \) is infinite. If \( n \geq 3 \), the so-called ‘full twist’ braid \( \Delta_n = (\sigma_1 \cdots \sigma_{n-1})^n \) generates the centre \( Z(B_n(S^2)) \) of \( B_n(S^2) \) and is the unique element of \( B_n(S^2) \) of order 2.

Our aim in this paper is to study the lower central and derived series of the braid groups of the sphere. Let us recall some definitions and notation concerning these series. If \( G \) is a group, then its lower central series \( \{ \Gamma_i(G) \}_{i \in \mathbb{N}} \) is defined inductively by \( \Gamma_1(G) = G \), and \( \Gamma_{i+1}(G) = [G, \Gamma_i(G)] \) for all \( i \in \mathbb{N} \), and its derived series \( \{ G^{(i)} \}_{i \in \mathbb{N}, j \in \{0\} } \) is defined inductively by \( G^{(0)} = G \), and \( G^{(i)} = [G^{(i-1)}, G^{(i-1)}] \) for all \( i \in \mathbb{N} \). One may check easily that \( \Gamma_i(G) \supseteq \Gamma_{i+1}(G) \) and \( G^{(i-1)} \supseteq G^{(i)} \) for all \( i \in \mathbb{N} \), and for all \( j \in \mathbb{N}, j > i \), \( \Gamma_j(G) \) (resp. \( G^{(j)} \)) is a normal subgroup of \( \Gamma_i(G) \) (resp. \( G^{(i)} \)). Notice that \( \Gamma_2(G) = G^{(1)} \) is the commutator subgroup of \( G \).

The Abelianisation of the group \( G \), denoted by \( G^{\text{Ab}} \), is the quotient \( G/\Gamma_2(G) \); the Abelianisation of an element \( g \in G \) is its \( \Gamma_2(G) \)-coset in \( G^{\text{Ab}} \). The group \( G \) is said to be perfect if \( G = G^{(1)} \), or equivalently if \( G^{\text{Ab}} = \{1\} \). Following P. Hall, for any group-theoretic property \( \mathcal{P} \), a group \( G \) is said to be residually \( \mathcal{P} \) if for any (non-trivial) element \( x \in G \), there exists a group \( H \) with the property \( \mathcal{P} \) and a surjective homomorphism \( \varphi: G \rightarrow H \) such that \( \varphi(x) \neq 1 \). It is well known that a group \( G \) is residually nilpotent (respectively residually soluble) if and only if \( \bigcap_{i \geq 1} \Gamma_i(G) = \{1\} \) (respectively \( \bigcap_{i \geq 0} G^{(i)} = \{1\} \)). If \( g, h \in G \), then \( [g, h] = ghg^{-1}h^{-1} \) will denote their commutator, and we shall use the notation \( g = h \) to mean that \( g \) and \( h \) commute.

The lower central series of groups and their successive quotients \( \Gamma_i/\Gamma_{i+1} \) are isomorphism invariants and have been widely studied using commutator calculus; in particular for free groups of finite rank \([36, 37]\). Falk and Randell, and independently Kolmo investigated the lower central series of the pure braid group \( \mathcal{P}_n \) and were able to conclude that \( \mathcal{P}_n \) is residually nilpotent \([15, 36]\). Falk and Randell also studied the lower central series of generalised pure braid groups \([16, 17]\).

Using the Reidemeister-Schreier rewriting process, Gorin and Lin obtained a presentation of the commutator subgroup of \( B_n \) for \( n \geq 3 \) \([32]\). For \( n \geq 5 \), they were able to infer that \( (B_n)^{(1)} = (B_n)^{(2)} \), and so conclude that \( (B_n)^{(1)} \) is perfect. From this it follows that \( \Gamma_2(B_n) = \Gamma_3(B_n) \); hence \( B_n \) is not residually nilpotent. If \( n = 3 \), then they showed that \( (B_3)^{(1)} \) is a free group of rank 2, while if \( n = 4 \), they proved that \( (B_4)^{(1)} \) is a semi-direct product of two free groups of rank 2. By considering the action, one may see that \( (B_4)^{(1)} \nsubseteq (B_4)^{(2)} \). The work of Gorin and Lin on these series was motivated by the study of almost periodic solutions of algebraic equations with almost periodic coefficients.

The above comments indicate that the study of the lower central and derived series of the braid groups of the sphere is an important problem in its own right, and it enables us to understand better the structure of such groups. But we are also motivated by the interesting question of the existence of a section, or ‘splitting problem’, for the following two short exact sequences of braid groups (notably for the case \( M = S^2 \)) obtained by considering the long exact sequences in homotopy of fibrations of the corresponding configuration spaces:

\[
1 \longrightarrow P_{m-n}(M \setminus \{x_1, \ldots, x_n\}) \overset{\partial}{\longrightarrow} P_m(M) \overset{p_n}{\longrightarrow} P_n(M) \longrightarrow 1,
\]

(1.3)
where $n \geq 3$ if $M$ is the 2-sphere $S^2$ [11, 14], $n \geq 2$ if $M$ is the real projective plane $\mathbb{R}P^2$ [44], and $n \geq 1$ otherwise [13], and where $p_*$ is the group homomorphism which geometrically corresponds to forgetting the last $m-n$ strings, and $i_*$ is inclusion (we consider $P_{m-n}(M \setminus \{x_1, \ldots, x_n\})$ to be the subgroup of $P_m(M)$ of pure braids whose last $n$ strings are vertical). This short exact sequence plays a central role in the study of surface braid groups. It was used for example to study mapping class groups [39], Vassiliev invariants for braid groups [31], as well as to obtain presentations for surface pure braid groups [5, 20, 23, 41].

We remark that if the above conditions on $n$ and $m$ are satisfied, then the existence of a section for $p_*$ is equivalent to that of a geometric section for the corresponding configuration spaces (cf. [22, 23]). If $M$ is the plane, then the fact that (1.3) splits for all $n \in \mathbb{N}$ implies that $P_n$ may be expressed as a repeated semi-direct product of free groups [2], which leads to a solution of the word problem for $P_n$ and $B_n$. The splitting problem has been studied for other surfaces besides the plane. Fadell and Neuwirth gave various sufficient conditions for the existence of a geometric section for $p$ in the general case [13]. For the sphere, it was known that there exists a section on the geometric level [14]. If $M$ is the 2-torus, then Birman exhibited an explicit algebraic section for (1.3) for $m = n+1$ and $n \geq 2$ [5]. However, for compact orientable surfaces without boundary of genus $g \geq 2$, she posed the question of whether the short exact sequence (1.3) splits. In [20], we provided a complete answer to this question:

**Theorem 1.1** ([20]). If $M$ is a compact orientable surface without boundary of genus $g \geq 2$, the short exact sequence (1.3) splits if and only if $n = 1$.

In the case of $\mathbb{R}P^2$, Van Buskirk showed that the exact sequence (1.3) splits if $m = 3$ and $n = 2$ [44]. We recently showed that this condition is also necessary, and thus answered a question posed by Van Buskirk in that paper:

**Theorem 1.2** ([25]). Let $M$ be the real projective plane $\mathbb{R}P^2$. Then the short exact sequence (1.3) splits if and only if $m = 3$ and $n = 2$.

In [23], we studied the short exact sequence (1.4) in the case $M = S^2$ of the sphere, and showed that if $m = 3$, then the short exact sequence (1.4) splits if and only if $n \equiv 0, 2 \pmod{3}$. Further, if $m \geq 4$ and the short exact sequence (1.4) splits, then there exist $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$ such that $n \equiv \varepsilon_1(m-1)(m-2) - \varepsilon_2 m(m-2) \pmod{m(m-1)(m-2)}$. An open question is whether this condition is also sufficient.

The study in this paper of the lower central and derived series of the braid groups of the sphere was motivated in part by the question of the existence of a section for the short exact sequences (1.3) and (1.4). To obtain a positive answer, it suffices of course to exhibit an explicit section (although this may be easier said than done!). However, and in spite of the fact that we possess presentations of surface braid...
groups, in general it is very difficult to prove directly that such an extension does not split. One of the main methods that we used to prove the non-splitting of \( (1.3) \) for \( n \geq 2 \) and of \( (1.3) \) for \( m \geq 4 \) was based on the following observation: let \( 1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1 \) be a split extension of groups, where \( K \) is a normal subgroup of \( G \), and let \( H \) be a normal subgroup of \( G \) contained in \( K \). Then the extension \( 1 \rightarrow K/H \rightarrow G/H \rightarrow Q \rightarrow 1 \) also splits. The condition on \( H \) is satisfied for example if \( H \) is an element of either the lower central series or the derived series of \( K \). In [20], considering the extension \( (1.3) \) with \( n \geq 3 \), we showed that it was sufficient to take \( H = \Gamma_2(K) \) to prove the non-splitting of the quotiented extension, and hence that of the full extension. In this case, the kernel \( K/\Gamma_2(K) \) is Abelian, which simplifies somewhat the calculations in \( G/H \). This was also the case in [23] for the extension \( (1.4) \) with \( m \geq 4 \). However, for the extension \( (1.3) \) with \( n = 2 \), it was necessary to go a stage further in the lower central series and take \( H = \Gamma_3(K) \). From the point of view of the splitting problem, it is thus helpful to know the lower central and derived series of the braid groups occurring in these group extensions. But as we indicated earlier, these series are of course interesting in their own right and help us to understand better the structure of surface braid groups.

1.2. Statement of the main results. This paper is organised as follows. In Section 2.1 we prove some general results regarding the splitting of the short exact sequence \( 1 \rightarrow \Gamma_2(B_n(S^2)) \rightarrow B_n(S^2) \rightarrow (B_n(S^2) \wedge \mathbb{Z})_2 \rightarrow 1 \), as well as homological conditions for the stabilisation of the lower central series of a group (Lemma 2.4).

In Section 2.2 we prove Theorem 1.3 which deals with the lower central series of \( B_n(S^2) \):

**Theorem 1.3.** For all \( n \geq 2 \), the lower central series of \( B_n(S^2) \) is constant from the commutator subgroup onwards: \( \Gamma_m(B_n(S^2)) = \Gamma_2(B_n(S^2)) \) for all \( m \geq 2 \). The subgroup \( \Gamma_2(B_n(S^2)) \) is as follows:

1. If \( n = 1, 2 \), then \( \Gamma_2(B_n(S^2)) = \{1\} \).
2. If \( n = 3 \), then \( \Gamma_2(B_n(S^2)) \cong \mathbb{Z}_3 \). Thus \( B_3(S^2) \cong \mathbb{Z}_3 \times \mathbb{Z}_4 \), the action being the non-trivial one.
3. If \( n = 4 \), then \( \Gamma_2(B_4(S^2)) \) admits a presentation of the following form:

**Generators:** \( g_1, g_2, g_3 \), where in terms of the usual generators of \( B_4(S^2) \), \( g_1 = \sigma_1^4 \sigma_2 \sigma_1^{-3}, g_2 = \sigma_1^3 \sigma_2 \sigma_1^{-4} \) and \( g_3 = \sigma_1 \sigma_1^{-1} \).

**Relations:**

\[
g_3^4 = 1,
\]

\[
[g_3^2, g_i] = 1 \quad \text{for} \quad i = 1, 2,
\]

\[
[g_3, g_2 g_1] = 1,
\]

\[
g_1^{-1} g_3^{-1} g_1 = g_2 g_3 g_2^{-1},
\]

\[
g_1^{-1} g_3^{-1} g_1 = g_1 g_3 g_1^{-1} g_3.
\]

Furthermore,

\[
\Gamma_2(B_4(S^2)) \cong Q_8 \rtimes F_2(a, b),
\]

where \( Q_8 = \langle x, y \mid x^2 = y^2, x y x^{-1} = y^{-1} \rangle \) is the quaternion group of order 8, and \( F_2(a, b) \) is the free group of rank 2 on two generators \( a \) and \( b \).
The following elements of $B_4(S^2)$ realise these subgroups: $x = g_3$, $y = g_1g_3g_1^{-1}$, $a = g_1$ and $b = g_2$. The action is given by:

$$\varphi(a) = y, \quad \varphi(a) = xy, \quad \varphi(b) = x.$$ 

(4) If $n \geq 4$, a presentation for $\Gamma_2(B_n(S^2))$ is as given in Proposition 4.4 (see Section 4.1).

The lower central series of $B_n(S^2)$ is thus completely determined. In particular, for all $n \geq 2$, the lower central series of $B_n(S^2)$ is constant from the commutator subgroup onwards, and $B_n(S^2)$ is residually nilpotent if and only if $n \leq 2$. The case $n = 4$ is particularly interesting: $\Gamma_2(B_4(S^2))$ is a semi-direct product of the quaternion group $Q_8$ of order 8 by the free group of rank 2. This may be compared with Gorin and Lin’s result for $\Gamma_2(B_4)$ [32]. Thus $B_4(S^2)$ contains an isomorphic copy of $Q_8$. We learnt that this inclusion had previously been studied by Thompson [43]. We then showed that for all $n \geq 3$, $B_n(S^2)$ contains an isomorphic copy of $Q_8$ of order 8 if and only if $n$ is even [24]. This leads naturally to the interesting problems of the classification of the finite and virtually cyclic subgroups of $B_n(S^2)$ and $B_n(\mathbb{R}P^2)$. We have recently classified the finite subgroups of $B_n(S^2)$ [25] and the finite and virtually cyclic subgroups of $P_n(\mathbb{R}P^2)$ [24]. We are currently pursuing the study of the virtually cyclic subgroups of $B_n(S^2)$, as well as the finite and virtually cyclic subgroups of $B_n(\mathbb{R}P^2)$ [30].

In Section 3 we study the derived series of $B_n(S^2)$. As in the case of $B_n$ [22], $(B_n(S^2))^{(1)}$ is perfect if $n \geq 5$; in other words, the derived series of $B_n(S^2)$ is constant from $(B_n(S^2))^{(1)}$ onwards. The cases $n = 1, 2, 3$ are straightforward, and the groups $B_n(S^2)$ are finite and soluble. In the case $n = 4$, we make use of the semi-direct product decomposition of $(B_4(S^2))^{(1)}$ obtained in Theorem 1.4. Proposition 3.3 describes the structure of the commutator subgroup of a general semi-direct product, and it seems to be a useful and interesting result. This enables us to show that from $(B_4(S^2))^{(4)}$ onwards, the derived series of $B_4(S^2)$ coincides with that of the free group of rank 2. We also determine some of the derived series quotients of $B_4(S^2)$.

**Theorem 1.4.** The derived series of $B_n(S^2)$ is as follows:

(1) If $n = 1, 2$, then $(B_n(S^2))^{(1)} = \{1\}$.

(2) If $n = 3$, then $(B_3(S^2))^{(1)} \cong \mathbb{Z}_3$ and $(B_3(S^2))^{(2)} = \{1\}$.

(3) Suppose that $n = 4$. Then:

(a) $(B_4(S^2))^{(1)} = \Gamma_2(B_4(S^2))$ is given by part (3) of Theorem 1.3; it is isomorphic to the semi-direct product $Q_8 \rtimes \mathbb{F}_2$. Further, the quotient $B_4(S^2)/(B_4(S^2))^{(1)}$ is isomorphic to $\mathbb{Z}_6$.

(b) $(B_4(S^2))^{(2)}$ is isomorphic to the semi-direct product $Q_8 \rtimes (\mathbb{F}_2)^{(1)}$, where $(\mathbb{F}_2)^{(1)}$ is the commutator subgroup of the free group $\mathbb{F}_2 = \mathbb{F}_2(a, b)$ of rank 2 on two generators $a, b$. The action of $(\mathbb{F}_2)^{(1)}$ on $Q_8$ is the restriction of the action of $\mathbb{F}_2(a, b)$ given in part (3) of Theorem 1.3. Further,

$$(B_4(S^2))^{(1)}/(B_4(S^2))^{(2)} \cong \mathbb{Z}^2 \quad \text{and} \quad B_4(S^2)/(B_4(S^2))^{(2)} \cong \mathbb{Z}^2 \times \mathbb{Z}_6,$$

where the action of the generator $\sigma$ of $\mathbb{Z}_6$ on $\mathbb{Z}^2$ is given by left multiplication by the matrix $\left( \begin{array}{cc} 0 & 1 \\ -1 & 1 \end{array} \right)$. 


(c) \((B_4(S^2))^{(3)}\) is a subgroup of \(P_4(S^2)\) isomorphic to the direct product \(\mathbb{Z}_2 \times (F_2)^{(2)}\). Further, 
\[ (B_4(S^2))^{(2)}/(B_4(S^2))^{(3)} \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (F_2)^{(1)}/(F_2)^{(2)}. \]
(d) \((B_4(S^2))^{(m)} \cong (F_2)^{(m-1)}\) for all \(m \geq 4\). Further, 
\[ (B_4(S^2))^{(3)}/(B_4(S^2))^{(4)} \cong \mathbb{Z}_2 \times (F_2)^{(2)}/(F_2)^{(3)}, \]
and for \(m \geq 4\),
\[ (B_4(S^2))^{(m)}/(B_4(S^2))^{(m+1)} \cong (F_2)^{(m-1)}/(F_2)^{(m)}. \]
(4) If \(n \geq 5\), then \((B_n(S^2))^{(2)} = (B_n(S^2))^{(1)}\), so \((B_n(S^2))^{(1)}\) is perfect. A presentation of \((B_n(S^2))^{(1)}\) is given in Proposition 4.1.

So the derived series of \(B_n(S^2)\) is thus completely determined (up to knowing the derived series of the free group \(F_2\) of rank 2; see Remark 4.1). In particular, \(B_n(S^2)\) is residually soluble if and only if \(n \leq 4\) (Corollary 3.2). Thus \(B_4(S^2)\) is residually soluble, but not residually nilpotent.

Finally in Section 4, for \(n \geq 4\), we give presentations of the commutator subgroups \(\Gamma_2(B_n(S^2))\) of the sphere braid groups, and in the case \(n = 4\), in Proposition 4.3 we derive the presentation of \(\Gamma_2(B_4(S^2))\) given in Theorem 1.3.

In a companion paper \(27\), we study the lower central and derived series of the class of braid groups of the finitely-punctured sphere. This class includes the Artin braid groups, the braid groups of the annulus, which are Artin groups of type \(B\) (for \(n = 2\)), and affine Artin groups of type \(\tilde{C}\) (for \(n = 3\)). We remark that since work on these two papers started, one of the authors in collaboration with P. Bellingeri and S. Gervais has undertaken the study of the lower central series of braid groups of orientable surfaces, with and without boundary, of genus \(g \geq 1\) \(4\), and that some of the techniques appearing in this paper were subsequently used there. It is worth stating the following result of \(4\) which contrasts somewhat with that obtained here for the sphere.

**Theorem 1.5** (4). Let \(M\) be a compact, connected orientable surface without boundary, of genus \(g \geq 1\), and let \(m \geq 3\). Then:

1. \(\Gamma_1(B_m(M))/\Gamma_2(B_m(M)) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}_2.\)
2. \(\Gamma_2(B_m(M))/\Gamma_3(B_m(M)) \cong \mathbb{Z}_{m-1+g}.\)
3. \(\Gamma_3(B_m(M)) = \Gamma_4(B_m(M)).\) Moreover, \(\Gamma_3(B_m(M))\) is perfect for \(m \geq 5\).

This implies that braid groups of compact, connected orientable surfaces without boundary may be distinguished by their lower central series (indeed by the first two lower central quotients). Notice that the lower central series stabilises one stage further, i.e. from \(\Gamma_3\) onwards, as opposed to \(\Gamma_2\) in the case of the sphere.

2. **The lower central series of \(B_n(S^2)\)**

The main aim of this section is to prove Theorem 1.3 which describes the lower central series of \(B_n(S^2)\). This will be carried out in Section 2.2. Before doing so, in Section 2.1 we state and prove some general results concerning the splitting of the short exact sequence (2.1) (Proposition 2.1), as well as homological conditions for the stabilisation of the lower central series of a group (Lemma 2.4).
2.1. Generalities. Let $n \in \mathbb{N}$. Let $B_n(S^2)$ denote the braid group of $S^2$ on $n$ strings, let $(B_n(S^2))^{\text{Ab}} = B_n(S^2)/\Gamma_2(B_n(S^2))$ denote Abelianisation of $B_n(S^2)$, and let $\alpha: B_n(S^2) \to (B_n(S^2))^{\text{Ab}}$ be the canonical projection. Then we have the following short exact sequence:

$$1 \to \Gamma_2(B_n(S^2)) \to B_n(S^2) \to (B_n(S^2))^{\text{Ab}} \to 1. \quad (2.1)$$

We first prove the following result which deals with the splitting of this short exact sequence.

**Proposition 2.1.** Let $n \in \mathbb{N}$. Then:

1. $(B_n(S^2))^{\text{Ab}} = B_n(S^2)/\Gamma_2(B_n(S^2)) \cong \mathbb{Z}_{2(n-1)}$.
2. The short exact sequence (2.1) splits if and only if $n$ is even, where the action on $\Gamma_2(B_n(S^2))$ by a generator of $\mathbb{Z}_{2(n-1)}$ is given by conjugation by $\sigma_1 \cdots \sigma_n - 2\sigma_{n-1}^2$.
3. If $n$ is odd, then $B_n(S^2)$ is not isomorphic to the semi-direct product of a subgroup $K$ by $\mathbb{Z}_{2(n-1)}$.

**Proof.** (1) This follows easily from the presentation \([1,2]\) of the group $B_n(S^2)$. The generators $\sigma_i$ of $B_n(S^2)$ are all identified by $\alpha$ to a single generator $\tilde{\sigma} = \alpha(\sigma_i)$ of $\mathbb{Z}_{2(n-1)}$.

(2) In order to construct a section, we consider the elements of $B_n(S^2)$ of order $2(n-1)$. According to Murasugi’s classification of the torsion elements of $B_n(S^2)$ \([8]\), these elements are precisely the conjugates of the elements of the form $(\sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2)^r$, where $r$ and $2(n-1)$ are coprime. Such an element projects to $\tilde{\sigma}^r$ whose order is $2(n-1)/\gcd(rn, 2(n-1))$. Since

$$\gcd(rn, 2(n-1)) = \gcd(n, 2(n-1)) = \gcd(n, 2),$$

the result follows from equation (2.1) and part (1).

(3) Let $n \in \mathbb{N}$ be even. We first prove the following lemma:

**Lemma 2.2.** Let $G$ be a group whose Abelianisation $G^{\text{Ab}}$ is Hopfian, i.e. $G^{\text{Ab}}$ is not isomorphic to any of its proper quotients. Suppose that there exists a group $H$ isomorphic to $G^{\text{Ab}}$, a normal subgroup $K$ of $G$, and a split short exact sequence $1 \to K \to G \to H \to 1$. Then $G \cong \Gamma_2(G) \times G^{\text{Ab}}$.

**Proof of Lemma 2.2.** Let $\alpha: G \to G^{\text{Ab}}$ denote the Abelianisation homomorphism, let $\xi: G \to H$ denote the homomorphism in the given short exact sequence, and let $s: H \to G$ be a section for $\xi$. Since $H \cong G/K$ is Abelian, it follows from standard properties of the commutator subgroup that $\Gamma_2(G) \subseteq K$. Hence we have the following commutative diagram:

$$\begin{array}{ccc}
1 & \to & \Gamma_2(G) \xrightarrow{\xi} & G \xrightarrow{\alpha} & G^{\text{Ab}} & \to & 1 \\
1 & \to & K \xrightarrow{s} & G \xrightarrow{\xi} & H & \to & 1.
\end{array}$$

This extends to a commutative diagram of short exact sequences by taking $\rho: G^{\text{Ab}} \to H$ defined by $\rho(y) = \xi(x)$ for all $y \in G^{\text{Ab}}$, where $x \in G$ is any element satisfying $\alpha(x) = y$. This homomorphism is well defined, and is surjective since $\xi$ and $\alpha$ are surjective. But $G^{\text{Ab}} \cong H$ by hypothesis, which implies that $\rho$ is an
isomorphism. Hence $\alpha = \rho^{-1} \circ \xi$, and $s \circ \rho$ is a section for $\alpha$, which proves the lemma. 

By taking $G = B_n(S^2)$ and $K = \mathbb{Z}_{2(n-1)}$ in the statement of Lemma 2.2 if $B_n(S^2)$ were a semi-direct product of $K$ with $H$, then this would contradict part (2). This completes the proof of Proposition 2.1.

Remark 2.3. If $n$ is even, let us consider the natural projection $p: \mathbb{Z}_{2(n-1)} \to \mathbb{Z}_{n-1}$. Then we have a short exact sequence:

$$1 \to \Gamma_n^2(B_n(S^2)) \to B_n(S^2) \to \mathbb{Z}_{(n-1)} \to 1,$$

where $\alpha^* = p \circ \alpha$, and $\Gamma_n^2(B_n(S^2))$ is the kernel of $\alpha^*$. It is not difficult to see that this short exact sequence splits: a section is given by sending the generator of $\mathbb{Z}_{n-1}$ to $(\sigma_1 \ldots \sigma_{n-2}\sigma_{n-1})^{2^r}$, where $2^r$ is the greatest power of 2 dividing $n$.

Let $G$ be a group which acts on a group $H$. Following [35, page 67], we may define the commutator subgroup with respect to this action by

$$(2.2) \ \Gamma_G(H) = \langle (g \ast h)kh^{-1}k^{-1} \mid g \in G, h, k \in H \rangle,$$

where $g \ast h$ denotes the action of $g$ on $h$. We say that the action is perfect if $\Gamma_G(H) = H$. Note that if $H$ is a normal subgroup of $G$, then $H \supseteq \Gamma_G(H) = [G, H] \supseteq [H, H]$ for the action of conjugation of $G$ on $H$. In particular, if $G = H$, then $\Gamma_G(H) = \Gamma_2(G)$ for the action of conjugation of $G$ on itself. If this action is perfect, then the group $G$ is perfect.

Lemma 2.4. Let $G$ be a group, and let $G^{Ab}$ be its Abelianisation. Consider the homomorphism $\delta: H_2(G, \mathbb{Z}) \to H_2(G^{Ab}, \mathbb{Z})$ induced by Abelianisation. Then

$$\Gamma_2(G)/\Gamma_3(G) \cong \text{Coker}(\delta) \cong H_0 \left( G^{Ab}, H_1(\Gamma_2(G), \mathbb{Z}) \right).$$

In particular:

1. $\Gamma_2(G) = \Gamma_3(G)$ if and only if $\delta$ is surjective.
2. If $H_2(G^{Ab}, \mathbb{Z})$ is trivial, then $\Gamma_n(G) = \Gamma_2(G)$ for all $n \geq 2$.
3. If either the action (by conjugation) of $G$ on $\Gamma_2(G)$ or the action (by conjugation) of $G^{Ab}$ on $H_1(\Gamma_2(G), \mathbb{Z})$ is perfect, then $\Gamma_n(G) = \Gamma_2(G)$ for all $n \geq 2$.

Proof. Recall that if $1 \to K \to G \to Q \to 1$ is an extension of groups, then we have a 6-term exact sequence due to Stallings [35, 12]:

$$(2.3) \ \ H_2(G) \to H_2(Q) \to K/[G, K] \to H_1(G) \to H_1(Q) \to 1.$$ 

Applying this to the short exact sequence

$$(2.4) \ \ 1 \to \Gamma_2(G) \to G \to G^{Ab} \to 1,$$

we obtain

$$H_2(G, \mathbb{Z}) \to H_2(G^{Ab}, \mathbb{Z}) \to \Gamma_2(G)/\Gamma_3(G) \to H_1(G, \mathbb{Z}) \to G^{Ab} \to 1.$$ 

But $H_1(G, \mathbb{Z}) \to G^{Ab}$ is an isomorphism, so this becomes

$$H_2(G, \mathbb{Z}) \to H_2(G^{Ab}, \mathbb{Z}) \to \Gamma_2(G)/\Gamma_3(G) \to 1.$$ 

Hence $\Gamma_2(G)/\Gamma_3(G) \cong \text{Coker}(\delta)$, which yields the first isomorphism. To obtain the second, we consider the Lyndon-Hochschild-Serre spectral sequence [35] applied to the short exact sequence (2.4), for which the relevant terms are $E^2_{(2,0)} = H_2(G^{Ab}, \mathbb{Z})$.
and $E^{2}_{(0,1)} = H_{0} (G^{\text{Ab}}, H_{1} (\Gamma_{2}(G), \mathbb{Z})).$ Since $H_{1}(G) = H_{1}(G^{\text{Ab}}),$ the differential $d_{2}: E^{2}_{(2,0)} \rightarrow E^{2}_{(0,1)}$ is surjective, with kernel $E^{\infty}_{(2,0)}$. From the general definition of the filtration of $H_{2}(G)$ given by the spectral sequence, we have a surjection $H_{2}(G) \rightarrow E^{\infty}_{(2,0)},$ and hence the following exact sequence:

$$H_{2}(G) \rightarrow E^{\infty}_{(2,0)} \rightarrow E^{2}_{(2,0)} \rightarrow E^{2}_{(0,1)} \rightarrow 1.$$ 

Hence $\text{Im} (\delta) = E^{\infty}_{(2,0)},$ and

$$\text{Coker} (\delta) = E^{2}_{(2,0)}/\text{Im} (\delta) \cong E^{2}_{(0,1)} = H_{0} (G^{\text{Ab}}, H_{1} (\Gamma_{2}(G), \mathbb{Z})).$$

as required. From the first isomorphism, one may check that part (1) is satisfied. Part (2) then follows easily.

To prove part (3), if the action by conjugation of $G$ on $\Gamma_{2}(G)$ is perfect, then $\Gamma_{G}(\Gamma_{2}(G)) = [G, \Gamma_{2}(G)] = \Gamma_{3}(G) = \Gamma_{2}(G),$ and the result is clear. Now let us consider the action of $G$ on $H_{1}(\Gamma_{2}(G)) = (\Gamma_{2}(G))^{\text{Ab}}$ given by conjugation, defined by $g \cdot \tilde{h} = \overline{ghg^{-1}},$ where $g, h \in G$ and $\overline{\cdot}$ denotes Abelianisation in $\Gamma_{2}(G).$ If $g \in \Gamma_{2}(G),$ then the induced action on $(\Gamma_{2}(G))^{\text{Ab}}$ is trivial, so the original action factors through $G^{\text{Ab}},$ and we obtain an action of $G^{\text{Ab}}$ on $(\Gamma_{2}(G))^{\text{Ab}}$ given by $\tilde{g} \cdot \tilde{h} = \overline{ghg^{-1}}$ ($\tilde{g}$ denotes the Abelianisation of $g$ in $G$). Suppose that this action is perfect, so that $\Gamma_{G^{\text{Ab}}}((\Gamma_{2}(G))^{\text{Ab}}) = (\Gamma_{2}(G))^{\text{Ab}}.$ Now

$$\Gamma_{G^{\text{Ab}}}((\Gamma_{2}(G))^{\text{Ab}}) = [G, \Gamma_{2}(G)]/[\Gamma_{2}(G), \Gamma_{2}(G)] = \Gamma_{3}(G)/[\Gamma_{2}(G), \Gamma_{2}(G)],$$

and since $\Gamma_{3}(G) \subseteq \Gamma_{2}(G),$ it follows that $\Gamma_{3}(G) = \Gamma_{2}(G),$ which implies the result. 

\[\square\]

**Remark 2.5.** The hypothesis of part (2) of the lemma holds for example if $G^{\text{Ab}}$ is cyclic. Recall that if $G^{\text{Ab}}$ is finitely generated, then this condition is also necessary: if $H$ is a finitely-generated Abelian group satisfying $H_{2}(H, \mathbb{Z}) = \{0\},$ then $H$ is cyclic.

### 2.2. The lower central series of $B_{n}(S^{2})$.

Now we come to the main result of this section.

**Theorem 2.3.** For all $n \geq 2$, the lower central series of $B_{n}(S^{2})$ is constant from the commutator subgroup onwards: $\Gamma_{m}(B_{n}(S^{2})) = \Gamma_{2}(B_{n}(S^{2}))$ for all $m \geq 2$. The subgroup $\Gamma_{2}(B_{n}(S^{2}))$ is as follows:

(1) If $n = 1, 2,$ then $\Gamma_{2}(B_{n}(S^{2})) = \{1\}.$

(2) If $n = 3$, then $\Gamma_{2}(B_{n}(S^{2})) \cong \mathbb{Z}_{3}$. Thus $B_{3}(S^{2}) \cong \mathbb{Z}_{3} \times \mathbb{Z}_{4},$ the action being the non-trivial one.

(3) If $n = 4$, then $\Gamma_{2}(B_{4}(S^{2}))$ admits a presentation of the following form:

**generators:** $g_{1}, g_{2}, g_{3},$ where in terms of the usual generators of $B_{4}(S^{2})$, $g_{1} = \sigma_{1}^{2} \sigma_{2} \sigma_{1}^{-3},$ $g_{2} = \sigma_{1}^{2} \sigma_{2} \sigma_{1}^{-4}$ and $g_{3} = \sigma_{3} \sigma_{1}^{-1}.$
relations:

(2.5) \[ g_3^4 = 1, \]
(2.6) \[ [g_3^3, g_1] = 1, \]
(2.7) \[ [g_3^2, g_2] = 1, \]
(2.8) \[ [g_3, g_2 g_1] = 1, \]
(2.9) \[ g_1^{-1} g_3^{-1} g_1 = g_2 g_3 g_2^{-1}, \]
(2.10) \[ g_1^{-1} g_3^{-1} g_1 = g_1 g_3 g_1^{-1} g_3. \]

Furthermore,

\[ \Gamma_2(B_4(S^2)) \cong Q_8 \rtimes F_2(a, b), \]

where \( Q_8 = \langle x, y \mid x^2 = y^2, x y x^{-1} = y^{-1} \rangle \) is the quaternion group of order 8, and \( F_2(a, b) \) is the free group of rank 2 on two generators \( a \) and \( b \). The following elements of \( B_4(S^2) \) realise these subgroups: \( x = g_3, y = g_1 g_3 g_1^{-1}, a = g_1 \) and \( b = g_2 \). The action is given by:

\[
\begin{align*}
\varphi(a)(x) &= y, \\
\varphi(a)(y) &= xy, \\
\varphi(b)(x) &= yx, \\
\varphi(b)(y) &= x.
\end{align*}
\]

(4) If \( n \geq 5 \), a presentation for \( \Gamma_2(B_n(S^2)) \) is given in Proposition 4.1.

Proof. The first part of the theorem, \( \Gamma_m(B_n(S^2)) = \Gamma_2(B_n(S^2)) \) for \( m \geq 2 \), follows from Lemma 2.3 and Remark 2.6.

Now let us consider the rest of the theorem.

(1) If \( n = 1, 2 \), then \( B_n(S^2) \cong \mathbb{Z}_n \), and the result follows easily.

(2) Let \( n = 3 \). Then \( B_3(S^2) \) is a ZS-metacyclic group (a group whose Sylow subgroups, commutator subgroup and commutator quotient group are all cyclic) of order 12 [14]. It follows from Proposition 2.111 that \( (B_3(S^2))^\text{Ab} \cong \mathbb{Z}_4 \), and hence \( \Gamma_2(B_3(S^2)) \cong \mathbb{Z}_3 \).

From Proposition 2.112, the short exact sequence (2.1) splits, so we have \( B_3(S^2) \cong \mathbb{Z}_3 \times \mathbb{Z}_4 \), and the action of the generator \( \tilde{\sigma} \) of \( (B_3(S^2))^\text{Ab} \) on the generator \( \rho \) of \( \mathbb{Z}_3 \) is given by \( \tilde{\sigma} \cdot \rho = \rho^{-1} \), i.e. the non-trivial action.

(3) Let \( n = 4 \). To obtain the given presentation of \( \Gamma_2(B_4(S^2)) \), one applies the Reidemeister-Schreier rewriting process to the short exact sequence (2.1). The calculations are deferred to Proposition 4.3 see Section 4.2.

Supposing this presentation to be correct, let us prove the second part of Theorem 1.3 that \( \Gamma_2(B_4(S^2)) \cong Q_8 \rtimes F_2(a, b) \). Consider the subgroup \( H \) of \( B_4(S^2) \) generated by \( x = g_3 \) and \( y = g_1 g_3 g_1^{-1} \). By equation (2.6), \( x^2 = g_3^2 = (g_1 g_3 g_1^{-1})^2 = y^2 \). Now

\[
y^{-1} = g_1 g_3^{-1} g_1^{-1} = \sigma_1^2 \sigma_2 \sigma_1^{-3} \cdot \sigma_1 \sigma_3^{-1} \cdot \sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-2} = \sigma_1^2 \sigma_2 \sigma_1 \sigma_3 \sigma_2 = \sigma_1^2 \sigma_2 \sigma_3 \sigma_1 \sigma_2,
\]

and

\[
y^{-1} x^{-1} = \sigma_1^2 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_3^{-1} \sigma_1 = \sigma_1^2 \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_3 \sigma_2 \sigma_1 = \sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_1,
\]

which is the half-twist of \( B_4(S^2) \). It was shown in [24] (see the first paragraph of the proof of the Theorem of that paper) that the two elements \( y^{-1} x^{-1} \) and \( y^{-1} \) together generate a subgroup of \( B_4(S^2) \) isomorphic to \( Q_8 \). Since these two elements generate
of obtained in Theorem 1.3, we shall be able to prove that the derived series of $3^36$ D. L. GONÇALVES AND J. GUASCHI

\[
\begin{aligned}
g_1xg_1^{-1} &= y, \\
g_1yg_1^{-1} &= x^{-1}y^{-1} = xy & \text{by equation (2.10) and the relations of } Q_8, \\
g_2xg_2^{-1} &= g_1^{-1}g_3^{-1}g_1 = yx & \text{by equations (2.9) and (2.10),}
\end{aligned}
\]

Since $y = g_1g_3g_1^{-1}$, this normality implies that $H$ is in fact the normal closure of $g_3$ in $\Gamma_2(B_4(S^2))$. Studying the given presentation of $\Gamma_2(B_4(S^2))$, we see that $\Gamma_2(B_4(S^2))/H$ is isomorphic to the free group $F_2(a, b)$ on two generators $a$ and $b$ (which are the $H$-cosets of $g_1$ and $g_2$). Thus the short exact sequence $1 \rightarrow H \rightarrow \Gamma_2(B_4(S^2)) \rightarrow F_2(a, b) \rightarrow 1$ splits, a section being given by sending $a$ to $g_1$ and $b$ to $g_2$. The action is then defined by the equations (2.11), which are those given in the statement of the theorem.

(4) Now suppose that $n \geq 5$. A presentation is given in Proposition 4.1. This completes the proof of Theorem 1.3.

\hfill \Box

3. THE DERIVED SERIES OF $B_n(S^2)$

In this section, we study the derived series of $B_n(S^2)$. The aim is to prove the following result, which shows that for all $n \neq 3, 4$, $(B_n(S^2))^{(1)}$ is perfect. The difficult case is $n = 4$, but using the semi-direct product structure of $(B_4(S^2))^{(1)}$ obtained in Theorem 1.3, we shall be able to prove that the derived series of $B_4(S^2)$ coincides from a certain point with that of the free group of rank 2. Before doing so, we recall the statement of Theorem 1.4 and then state and prove Proposition 3.3, which describes the commutator subgroup of a general semi-direct product.

**Theorem 1.4** The derived series of $B_n(S^2)$ is as follows:

1. If $n = 1, 2$, then $(B_n(S^2))^{(1)} = \{1\}$.
2. If $n = 3$, then $(B_3(S^2))^{(1)} \cong \mathbb{Z}_2$ and $(B_3(S^2))^{(2)} = \{1\}$.
3. Suppose that $n = 4$. Then:
   a. $(B_4(S^2))^{(1)} = \Gamma_2(B_4(S^2))$ is given by part $\mathcal{B}$ of Theorem 1.3; it is isomorphic to the semi-direct product $Q_8 \rtimes F_2$. Further, the quotient $B_4(S^2)/(B_4(S^2))^{(1)}$ is isomorphic to $\mathbb{Z}_6$.
   b. $(B_4(S^2))^{(2)}$ is isomorphic to the semi-direct product $Q_8 \rtimes (F_2)^{(1)}$, where $(F_2)^{(1)}$ is the commutator subgroup of the free group $F_2(a, b)$ of rank 2 on two generators $a, b$. The action of $(F_2)^{(1)}$ on $Q_8$ is the restriction of the action of $F_2(a, b)$ given in part $\mathcal{B}$ of Theorem 1.3.
   Further,
   
   \[
   (B_4(S^2))^{(1)}/(B_4(S^2))^{(2)} \cong \mathbb{Z}^2 \quad \text{and} \quad B_4(S^2)/(B_4(S^2))^{(2)} \cong \mathbb{Z}^2 \rtimes \mathbb{Z}_6,
   \]
   where the action of the generator $\bar{e}$ of $\mathbb{Z}_6$ on $\mathbb{Z}^2$ is given by left multiplication by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$.
   c. $(B_4(S^2))^{(3)}$ is a subgroup of $P_4(S^2)$ isomorphic to the direct product $\mathbb{Z}_2 \times (F_2)^{(2)}$. Further,
   
   \[
   (B_4(S^2))^{(2)}/(B_4(S^2))^{(3)} \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes (F_2)^{(1)}/(F_2)^{(2)}.
   \]
Proposition 3.3. Which describes the commutator subgroup of a semi-direct product. This result

Corollary 3.2. On \( H \) to element of the form \( \phi(h) \cdot w \), where \( h \) runs over \( F_2(a,b) \). Then a basis is given by the set of elements of the form \( w_g[a,b]w_g^{-1} \), where \( g \) runs over \( F_2(a,b) \). For example, the set \( \{ a^pb^q[a,b]b^{-q}a^{-r} \} \) is a basis of \( F_2(a,b) \). Since \( F_2(a,b) \) is residually nilpotent and \( (F_2(a,b))^{(i)} \subseteq \Gamma_i(F_2(a,b)) \), it follows that \( \bigcap_{i \geq 0} (F_2(a,b))^{(i)} = \{ 1 \} \) and \( F_2(a,b) \) is residually soluble.

We easily obtain the following corollary of Theorem 1.4.

Corollary 3.2. Let \( n \in \mathbb{N} \). Then \( B_n(S^2) \) is residually soluble if and only if \( n \leq 4 \).

Proof of Corollary 3.2. Recall that a group \( G \) is residually soluble if and only if \( \bigcap_{i \geq 0} G^{(i)} = \{ 1 \} \). If \( n = 1, 2, 3 \), this is obvious, and if \( n = 4 \), the residual solubility of \( B_4(S^2) \) follows from that of \( F_2(a,b) \). For \( n \geq 5 \), the result also follows easily, since \( (B_n(S^2))^{(1)} \) is non-trivial.

Before proving Theorem 1.4, let us state and prove the following proposition which describes the commutator subgroup of a semi-direct product. This result will be used frequently throughout the rest of this paper.

Proposition 3.3. Let \( G, H \) be groups, and let \( \varphi: G \to \text{Aut}(H) \) be an action of \( G \) on \( H \). Let \( \tilde{H} \) be the subgroup of \( H \) generated by the elements of the form \( \varphi(g)(h) \cdot h^{-1} \), where \( g \in G \) and \( h \in H \). Let \( L \) be the subgroup of \( H \) generated by \( \Gamma_2(H) \) and \( \tilde{H} \). Then \( \varphi \) induces an action (also denoted by \( \varphi \)) of \( \Gamma_2(G) \) on \( L \), and \( L \rtimes_{\varphi} \Gamma_2(G) = \Gamma_2(H \rtimes_{\varphi} G) \). In particular, \( \Gamma_2(H \rtimes_{\varphi} G) \) is the subgroup generated by \( \Gamma_2(H), \Gamma_2(G) \) and \( \tilde{H} \).

Remark 3.4. We claim that \( L \) is none other than the commutator subgroup \( \Gamma_G(H) \) defined by equation (2.2) with respect to the given action. To see this, recall that \( \Gamma_G(H) \) is the subgroup of \( H \) generated by the elements of the form \( \varphi(g)(h) \cdot k^{-1}h^{-1} \), where \( g \in G \) and \( h, k \in H \). Taking \( g = e \) (respectively \( k = e \)), it follows that \( \Gamma_G(H) \supseteq \Gamma_2(H) \) (respectively \( \Gamma_G(H) \supseteq \tilde{H} \)), and hence \( L \subseteq \Gamma_G(H) \). Conversely, \( \varphi(g)(h) \cdot k^{-1}h^{-1} = \varphi(g)(h)h^{-1} \cdot khh^{-1}k^{-1} \in L \), so \( \Gamma_G(H) \subseteq L \), and the claim is
proved. Note further that if \( h' \in H \), then there exists \( h'' \in H \) such that \( \varphi(g)(h'') = h' \), so
\[
\varphi(g)(h'' h) h^{-1} = \varphi(g)(h'') h h^{-1} = \varphi(g)(h'' h) h^{-1} \cdot (\varphi(g)(h'')) h^{-1} = \varphi(g)(h'') h^{-1}.
\]
It follows that \( \tilde{H} \) and \( L \) are normal in \( H \). In particular, \( \Gamma_G(H) \) is normal in \( H \).

**Proof of Proposition 3.3** From now on, we shall identify each subgroup \( H_1 \) of \( H \) (respectively each subgroup \( G_1 \) of \( G \)) with the corresponding subgroup of the form \( \{ (h, 1) \mid h \in H_1 \} \) (respectively \( \{ (1, g) \mid g \in G_1 \} \)) of \( \Gamma \times \varphi G \) without further comment. The group operation in \( H \times \varphi G \) shall be written as
\[
(h, g) \cdot (h', g') = (h \cdot \varphi(g)(h'), g g'), \text{ where } (h, g), (h', g') \in H \times \varphi G.
\]
The subgroup \( L \) is normal in \( H \) by Remark 3.4. Let us show that \( \varphi \) induces an action (also denoted by \( \varphi \)) of \( G \) on \( L \). Let \( g \in G \). Since \( \varphi(g)(h_1, h_2) = \varphi(h_1, h_2) \in \Gamma_2(H) \) for all \( h_1, h_2 \in H \), and
\[
\varphi(g)(\varphi(g')(h) h^{-1}) = \varphi(g g')(h) h^{-1} \cdot (\varphi(g') h)^{-1} \in \tilde{H}
\]
for all \( h \in H \) and \( g' \in G \), it follows that \( \varphi(g)(L) \subseteq L \). Clearly \( \varphi(g) \) is injective. The surjectivity of \( \varphi(g) \) (restricted to \( L \)) may be deduced from the following observations:

1. if \( i = 1, 2 \) and \( h'_i \in H \), then there exists \( h_i \in H \) such that \( \varphi(g)(h_i) = h'_i \), and hence \( \varphi(g)([h_1, h_2]) = [h'_1, h'_2] \).
2. if \( g' \in G \) and \( h, h' \in H \), then
\[
\varphi(g) (\varphi(g^{-1} g')(h) h^{-1} \cdot h (\varphi(g^{-1})(h^{-1}) h) h^{-1}) = \varphi(g')(h) h^{-1}.
\]
Thus \( \varphi \) induces an action (also denoted by \( \varphi \)) of \( \Gamma_2(G) \) on \( L \), and \( L \times \varphi \Gamma_2(G) \) is a subgroup of \( H \times \varphi G \).

Clearly any element of \( \Gamma_2(H) \) (respectively \( \Gamma_2(G) \)) may be written as an element of \( \Gamma_2(H \times \varphi G) \). Further, if \( g \in G \) and \( h \in H \), then
\[
[(1, g), (h, 1)] = (\varphi(g)(h), 1) \cdot (h^{-1}, 1) = (\varphi(g)(h) h^{-1}, 1),
\]
and thus every element of \( \tilde{H} \) may be written as an element of \( \Gamma_2(H \times \varphi G) \). This proves that \( L \times \varphi \Gamma_2(G) \subseteq \Gamma_2(H \times \varphi G) \).

To see the converse, note that the commutator of the elements \( (h_1, g_1), (h_2, g_2) \in H \times \varphi G \) may be written as
\[
[(h_1, g_1), (h_2, g_2)] = (h_1 \cdot \varphi(g_1)(h_2), \varphi(g_1 g_2 g_1^{-1})(h_1^{-1}) \cdot \varphi([g_1, g_2])(h_2^{-1}), [g_1, g_2]).
\]
The second factor belongs to \( \Gamma_2(G) \). The first factor is of the form
\[
\varphi(g_1)(h_2 h_1^{-1} h_2^{-1} \cdot h g h_1 (\varphi(g_1 g_2 g_1^{-1})(h_1^{-1}) h_1) h_1^{-1} h_2^{-1} \cdot h_2 (\varphi(g_1, g_2))(h_2^{-1}) h_2^{-1},
\]
which is a product of elements of \( L \). Hence \( \Gamma_2(H \times \varphi G) \subseteq L \times \varphi \Gamma_2(G) \), and the proposition follows.

We now prove the main result of this section.

**Proof of Theorem 1.4** Cases (1) and (2) follow directly from Theorem 1.3.
Now consider case (1), i.e. $n \geq 5$. Let $H \subseteq (B_n(S^2))^{(1)}$ be a normal subgroup of $B_n(S^2)$ such that $A = (B_n(S^2))^{(1)}/H$ is Abelian (notice that this condition is satisfied if $H = (B_n(S^2))^{(2)}$). Let
\[
\begin{cases}
\pi: B_n(S^2) \to B_n(S^2)/H, \\
\beta \mapsto \overline{\beta}
\end{cases}
\]
denote the canonical projection. Then the Abelianisation homomorphism
\[
\alpha: B_n(S^2) \to (B_n(S^2))^{\text{Ab}}
\]
of Section 2 factors through $B_n(S^2)/H$, i.e. there exists a (surjective) homomorphism $\hat{\alpha}: B_n(S^2)/H \to (B_n(S^2))^{\text{Ab}}$ satisfying $\alpha = \hat{\alpha} \circ \pi$. So we have the following short exact sequence:
\[
1 \longrightarrow A \longrightarrow B_n(S^2)/H \longrightarrow (B_n(S^2))^{\text{Ab}} \longrightarrow 1.
\]
Now $\sigma_1, \ldots, \sigma_{n-1}$ generate $B_n(S^2)/H$, but since $\alpha(\sigma_i) = \alpha(\sigma_1)$ for $1 \leq i \leq n-1$, it follows that $\hat{\alpha}(\overline{\sigma_i}) = \hat{\alpha}(\overline{\sigma_1})$, and so there exists $t_i \in A$ such that $\overline{\sigma_i} = t_i \overline{\sigma_1}$.

We now apply $\pi$ to each of the relations of equation (12) of $B_n(S^2)$. First suppose that $3 \leq i \leq n-1$. Since $\sigma_i$ commutes with $\sigma_1$, we have that $\overline{\sigma_i} \cdot t_1 \overline{\sigma_1} = t_1 \overline{\sigma_i} \cdot \overline{\sigma_1}$, and hence $t_i$ commutes with $\overline{\sigma_1}$.

Now let $4 \leq i \leq n-1$. Since $\sigma_i$ commutes with $\sigma_2$, we obtain $t_1 \overline{\sigma_1} \cdot t_2 \overline{\sigma_1} = t_2 \overline{\sigma_1} \cdot t_1 \overline{\sigma_1}$. Since $A$ is Abelian, it follows from the previous paragraph that $t_2$ commutes with $\overline{\sigma_1}$. Applying this to the image of the relation $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$, under $\pi$, we see that $t_2 = t_1 = 1$, and hence $t_2 = 1$.

Next, if $i \geq 2$, then the relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ implies that $t_i = t_{i+1}$, and so $t_2 = \cdots = t_{n-1} = 1$. Hence $\overline{\sigma_1} = \overline{\sigma_2} = \cdots = \overline{\sigma_{n-1}}$. Thus $B_n(S^2)/H$ is cyclic, generated by $\overline{\sigma_1}$, and finite of order not greater than $2(n-1)$, because the surface relation $\sigma_1 \cdots \sigma_{n-2} \sigma_{n-1} \sigma_{n-2} \cdots \sigma_1 = 1$ projects to $\overline{\sigma_1}^{2(n-1)} = 1$. Since $\hat{\alpha}$ is surjective and $(B_n(S^2))^{\text{Ab}} \cong \mathbb{Z}_{2(n-1)}$, we conclude that $\hat{\alpha}$ is an isomorphism, so $B_n(S^2)/H \cong \mathbb{Z}_{2(n-1)}$, and $A = (B_n(S^2))^{(1)}/H$ is trivial. In particular
\[
(B_n(S^2))^{(2)} = [(B_n(S^2))^{(1)}, (B_n(S^2))^{(1)}] = (B_n(S^2))^{(1)},
\]
so $(B_n(S^2))^{(1)}$ is perfect.

Now consider case (3), so $n = 4$. Recall that part (3a) was proved in Theorem 1.3 and Proposition 2.1. To obtain $(B_4(S^2))^{(2)}$, it suffices to observe that for the action of $F_2(a,b)$ on $Q_8$, the subgroup $\hat{Q}_8$ defined in Proposition 3.3 is $Q_8$ (which is the case, since by Theorem 1.3 [3], $\varphi(b)(x)x^{-1} = y$ and $\varphi(a)(y)y^{-1} = x$). So $(B_4(S^2))^{(2)}$ is generated by $Q_8$ and $(F_2)^{(1)}$, $(B_4(S^2))^{(2)} \cong Q_8 \rtimes (F_2)^{(1)}$, and the action is the restriction of that of $F_2(a,b)$ on $Q_8$, which proves the first part of (3).

To determine $(B_4(S^2))^{(3)}$, we first have to describe the subgroup $\hat{Q}_8$ for the action of $(F_2)^{(1)}$ on $Q_8$. By Theorem 1.3 [3], if $B = [a,b] \in (F_2(a,b))^{(1)}$, then the automorphism $\varphi(B)$ satisfies $\varphi(B)(z) = x^2 \cdot z$ for $z \in \{x, y\}$ (recall that $x^2 = y^2$).

Since $(F_2(a,b))^{(1)}$ is the subgroup of $F_2(a,b)$ normally generated by $B$ and the centre $\langle x^2 \rangle$ of $Q_8$ is invariant under $\text{Aut}(Q_8)$, it follows that $\hat{Q}_8 = \langle x^2 \rangle$. So $(B_4(S^2))^{(3)}$ is isomorphic to the semi-direct product of $\mathbb{Z}_2$ by $(F_2)^{(2)}$. But the action is trivial, and so the product is direct. This proves the first part of (3).

For $m \geq 4$, the subgroup $(B_4(S^2))^{(m)}$ is clear from the description of $(B_4(S^2))^{(3)}$, and hence we obtain the first part of (3).
For several values of \( m \), we now analyse the quotients \( B_4(S^2)/(B_4(S^2))^{(m)} \) and \( (B_4(S^2))^{(m-1)}/(B_4(S^2))^{(m)} \). For the quotient \( B_4(S^2)/(B_4(S^2))^{(m)} \), we shall consider the case \( m = 2 \) (the case \( m = 1 \) is given by Proposition 2.11). For \( (B_4(S^2))^{(m-1)}/(B_4(S^2))^{(m)} \), we consider the cases \( m \geq 2 \) (the case \( m = 1 \) was considered in Proposition 2.11). If \( m > 4 \), the problem reduces to the corresponding problem for the free group on two generators.

We adopt the notation used above in the case \( n \geq 5 \), and again we suppose that \( H \subseteq (B_n(S^2))^{(1)} \) is a normal subgroup of \( B_n(S^2) \) such that \( A = (B_n(S^2))^{(1)}/H \) is Abelian. So we have a short exact sequence:

\[
1 \longrightarrow A \longrightarrow B_4(S^2)/H \xrightarrow{\alpha} \left( B_4(S^2) \right)^{Ab} \longrightarrow 1.
\]

Now \( \sigma_1, \sigma_2, \sigma_3 \) generate \( B_4(S^2)/H \). As above, for \( i = 2, 3 \) we set \( \sigma_i = t_i \sigma_1 \), where \( t_i \in A \), and we apply \( \pi \) to the relations of \( B_4(S^2) \). The fact that \( \sigma_1 \) commutes with \( \sigma_3 \) implies that \( \sigma_3 \) commutes with \( \sigma_1 \). The relation \( \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \) implies that

\[
\sigma_1 \sigma_2 \sigma_1^{-1} = \sigma_2 \cdot \sigma_1^{-2} \sigma_2 \sigma_1^{-2}.
\]

Similarly, the relation \( \sigma_2 \sigma_3 \sigma_2 = \sigma_3 \sigma_2 \sigma_3 \) implies that \( \sigma_3 \sigma_2 \sigma_3 = \sigma_2 \sigma_3 \sigma_2 \), since \( A \) is Abelian and \( t_2 \) commutes with \( \sigma_1 \). Thus \( t_2 \sigma_3^{-2} t_2 = t_3 \sigma_2^{-1} t_2 \sigma_1^{-1} \) (3.1), and so \( t_2 = 1 \). We conclude that \( B_4(S^2)/H \) is generated by \( \sigma_1 \) and \( t_2 \).

Finally, we consider the image of the surface relation under \( \pi \). Using equation (3.1), note that first that

\[
\sigma_1^3 t_2 \sigma_1^{-3} = \sigma_1(t_2^{-1} \cdot \sigma_1 t_2 \sigma_1^{-1} \cdot \sigma_1^{-1} t_2 \sigma_1^{-2} - \sigma_1^{-1} t_2 \sigma_1^{-1} = t_2^{-1} \sigma_1^{-1} \cdot t_2^{-1} \sigma_1 t_2 \sigma_1^{-1} = t_2^{-1},
\]

since \( A \) is normal and Abelian. Thus \( \sigma_1 \sigma_2 \sigma_3 \sigma_2 \sigma_1 = 1 \) implies that

\[
1 = \sigma_1 \cdot t_2 \sigma_1 \cdot \sigma_1^3 \cdot t_2 \sigma_1 \cdot \sigma_1 = \sigma_1 t_2 \sigma_1^{-1} \cdot \sigma_1 (\sigma_1 t_2 \sigma_1^{-3} \sigma_1^{-1} \cdot \sigma_1 \sigma_1 \sigma_1^4
\]

from equation (3.2).

Recall that \( \Gamma_2(B_4(S^2)) \) is the normal subgroup of \( B_4(S^2) \) generated by the commutators of the generators of \( B_4(S^2) \). Hence the normal subgroup \( B_4(S^2)/H \) generated by \( \sigma_1, t_2 \sigma_1^{-1} t_2^{-1} \) yields \( t_2^{-1} \). Further,

\[
\sigma_1 (\sigma_1 t_2 \sigma_1^{-1} t_2^{-1}) \sigma_1^{-1} = \sigma_1^2 t_2 \sigma_1^{-2} \cdot \sigma_1 t_2^{-1} \sigma_1^{-1} = t_2^{-1}
\]

from equation (3.1), and since \( \sigma_1^2 (\sigma_1 t_2 \sigma_1^{-1} t_2^{-1}) \sigma_1^{-2} = \sigma_1 t_2^{-1} \sigma_1^{-2} \), it follows that \( A \) is the Abelian group generated by \( \sigma_1 t_2 \sigma_1^{-1} t_2^{-1}, t_2 \) and \( \sigma_1 t_2 \sigma_1^{-1} \), and thus by \( t_2 \) and \( \sigma_1 t_2 \sigma_1^{-1} \).

Let \( \tilde{\sigma} = \alpha(\sigma_1) \) denote the generator of \( (B_4(S^2))^{Ab} \). Let \( M = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \); notice that \( M \) is of order 6. We now let \( (B_4(S^2))^{Ab} \approx \mathbb{Z}_6 \) act on \( \mathbb{Z}^2 \) as follows:

\[
\tilde{\sigma} \cdot \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = M \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_2 \\ X_2 - X_1 \end{pmatrix}.
\]
and so we may form the associated semi-direct product $\mathbb{Z}^2 \rtimes \mathbb{Z}_6$. We now consider the following homomorphism:

$$\psi : B_4(S^2) \to \mathbb{Z}^2 \times \mathbb{Z}_6,$$

$$\begin{align*}
\sigma_1, \sigma_3 &\mapsto \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \\
\sigma_2 &\mapsto \left( \begin{array}{c} 1 \\ 0 \end{array} \right).
\end{align*}$$

An easy calculation shows that $\psi$ preserves the relations of $B_4(S^2)$, and so is a well-defined homomorphism. Further, since $\psi(\sigma_1) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \psi(\sigma_2\sigma_3^{-1}) = \left( \begin{array}{c} 1 \\ 0 \end{array} \right)$ and $\psi([\sigma_1, \sigma_2]) = \left( \begin{array}{c} 0 \\ 1 \end{array} \right)$, $\psi$ is surjective.

Now let $H = (B_4(S^2))^{(2)}$, and let $\delta : \mathbb{Z}^2 \times \mathbb{Z}_6 \to \mathbb{Z}_6$ denote the projection onto the second factor. Since $\mathbb{Z}_6$ is Abelian, it follows that $\delta(\psi(x))$ is trivial for all $x \in (B_4(S^2))^{(1)}$, so $\psi(x)$ belongs to the $\mathbb{Z}^2$-factor. Hence,

$$H = [(B_4(S^2))^{(1)}, (B_4(S^2))^{(1)}] \subseteq \ker \psi,$$

and thus $\psi$ factors through $A = B_4(S^2)/H$, inducing a (surjective) homomorphism $\hat{\psi} : B_4(S^2)/H \to \mathbb{Z}^2 \times \mathbb{Z}_6$. From the following commutative diagram of short exact sequences,

$$
\begin{array}{c}
1 \to A = \Gamma_2(B_4(S^2))/H \to B_4(S^2)/H \xrightarrow{\hat{\psi}} (B_4(S^2))^{\text{Ab}} \to 1 \\
1 \to \mathbb{Z}^2 \xrightarrow{\psi} \mathbb{Z}^2 \times \mathbb{Z}_6 \xrightarrow{\delta} \mathbb{Z}_6 \to 1,
\end{array}
$$

the surjectivity of $\hat{\psi}$ implies that of $\hat{\psi}|_A : A \to \mathbb{Z}^2$. But $A$ is an Abelian group generated by $\{t_2, \sigma_2\sigma_1^{-1}\}$, so $\hat{\psi}|_A$ is an isomorphism, and by the 5-Lemma, $\hat{\psi}$ is too. Hence,

$$(B_4(S^2))^{(1)}/(B_4(S^2))^{(2)} \cong \mathbb{Z}^2 \text{ and } B_4(S^2)/(B_4(S^2))^{(2)} \cong \mathbb{Z}^2 \times \mathbb{Z}_6.$$ 

In fact the first of these two equations may be obtained directly since we know that $(B_4(S^2))^{(1)} \cong Q_8 \times F_2$, and $(B_4(S^2))^{(2)}$ is isomorphic to the subgroup $Q_8 \times (F_2)^{(1)}$ of $Q_8 \times F_2$, so $(B_4(S^2))^{(1)}/(B_4(S^2))^{(2)} \cong F_2/(F_2)^{(1)} \cong \mathbb{Z}^2$. Similarly, $(B_4(S^2))^{(2)}/(B_4(S^2))^{(3)} \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (F_2)^{(1)}/(F_2)^{(2)}$, $(B_4(S^2))^{(3)}/(B_4(S^2))^{(4)} \cong \mathbb{Z}_2 \times (F_2)^{(2)}/(F_2)^{(3)}$, and for $m \geq 4$,

$$(B_4(S^2))^{(m)}/(B_4(S^2))^{(m+1)} \cong (F_2)^{(m-1)}/(F_2)^{(m)}.$$ 

This proves the remaining parts of $(\mathfrak{B})$, and thus completes the proof of Theorem 1.3

\hfill $\Box$

4. Presentations for $\Gamma_2(B_n(S^2))$, $n \geq 4$

In Section 4.1 we derive a general presentation obtained using the Reidemeister-Schreier rewriting process. In Section 4.2 we consider the case $n = 4$, and obtain the presentation given in Theorem 1.3(3).
4.1. A general presentation of $\Gamma_2(B_n(S^2))$ for $n \geq 4$.

**Proposition 4.1.** Let $n \geq 4$. The following constitutes a presentation of the group $\Gamma_2(B_n(S^2))$:

**Generators:**

\[ w = \sigma_1^{2n-2}, \]
\[ u_1 = \sigma_2\sigma_1^{-1}, \quad u_2 = \sigma_1\sigma_2\sigma_1^{-2}, \quad \ldots, \quad u_{2n-2} = \sigma_1^{2n-3}\sigma_2\sigma_1^{-(2n-2)}, \]
\[ v_1 = \sigma_3\sigma_1^{-1}, \quad \ldots, \quad v_{n-3} = \sigma_{n-1}\sigma_1^{-1}. \]

**Relations:**

\[ v_iv_j = v_jv_i \text{ if } |i - j| \geq 2 \text{ and } 1 \leq i, j \leq n - 3, \]
\[ v_iv_{i+1}v_i = v_{i+1}v_iv_{i+1} \text{ for all } 1 \leq i \leq n - 4, \]
\[ w \overset{=} \Rightarrow v_i, \]
\[ u_iv_ju_i^{-1}v_j^{-1} = 1 \text{ for } j \geq 2 \text{ and } i = 1, \ldots, 2n - 3, \]
\[ u_{2n-2}v_jwv_i^{-1}w^{-1}v_j^{-1} = 1 \text{ for } 2 \leq j \leq n - 3, \]
\[ u_iu_{i+1}v_iu_i^{-1}v_i^{-1} = 1 \text{ for } i = 1, \ldots, 2n - 4, \]
\[ u_{2n-3}v_iwv_i^{-1}w^{-1}v_i^{-1} = 1, \]
\[ u_{2n-2}v_iwv_i^{-1}w^{-1}v_i^{-1} = 1, \]
\[ u_{i+1}u_{i+2}v_i^{-1} = 1 \text{ for all } i = 1, \ldots, 2n - 4, \]
\[ u_{2n-2}u_iwv_i^{-1}w^{-1}u_{2n-3} = 1, \]
\[ wu_1u_2w^{-1}u_{2n-2} = 1, \]
\[ u_2v_1\cdots v_{n-4}v_{n-3}^2v_{n-4}\cdots v_1u_{2n-3}w = 1, \]
\[ u_3(v_1\cdots v_{n-4}v_{n-3}^2v_{n-4}\cdots v_1)u_{2n-2}w = 1, \]
\[ u_i(v_1\cdots v_{n-4}v_{n-3}^2v_{n-4}\cdots v_1)wu_{i-3} = 1 \text{ for } i = 4, \ldots, 2n - 2, \]
\[ u_1(v_1\cdots v_{n-4}v_{n-3}^2v_{n-4}\cdots v_1)u_{2n-4}w = 1. \]

**Remark 4.2.** The above presentation may be simplified, notably in the cases where $n = 4, 5$, and may be used to show that $\Gamma_2(B_n(S^2))$ is perfect for $n \geq 5$.

In what follows, we shall denote by equation $(m_i)$ the equation $(m)$ of the above system for the parameter value $i$.

**Proof.** Taking the presentation (4.2) of $B_n(S^2)$ and the set $\{1, \sigma_1, \sigma_1^2, \ldots, \sigma_1^{2n-3}\}$ as a Schreier transversal, we apply the Reidemeister-Schreier rewriting process to the following short exact sequence:

\[ 1 \longrightarrow \Gamma_2(B_n(S^2)) \longrightarrow B_n(S^2) \longrightarrow (B_n(S^2))^{Ab} \longrightarrow 1. \]

As generators of $\Gamma_2(B_n(S^2))$, we obtain $w = \sigma_1^{2n-2}$, $\sigma_1^2\sigma_1^{-1}(j+1)$, and $\sigma_1^{2n-3}\sigma_i$, where $2 \leq i \leq n - 1$ and $0 \leq j \leq 2n - 4$. We replace the latter by $\sigma_1^{2n-3}\sigma_i^{-1}w^{-1} = \sigma_1^{2n-3}\sigma_i\sigma_1^{-(2n-2)}$. Now turning to the relations, if $i \geq 3$, then for $j = 0, \ldots, 2n - 4$, the relator $\sigma_1\sigma_i\sigma_1^{-1}\sigma_i^{-1}$ of $B_n(S^2)$ gives rise to relators

\[ \sigma_1\sigma_i\sigma_1^{-1}\sigma_i^{-1} = \sigma_i^j\sigma_i^{-(2n-2)}\sigma_i^{j+1}\sigma_i^{-1}\sigma_i^{-j}. \]
of $\Gamma_2(B_n(S^2))$, so

$$\sigma_i^{j+1}\sigma_i^{-1} = \sigma_i^j\sigma_i^{-1} = \sigma_i^{-1} = v_{i-2}.$$ 

If $j = 2n - 3$, then we have a relator of the form

$$\sigma_1^{2n-3}\sigma_1\sigma_1^{-1}\sigma_1^{-1} = \sigma_1^{2n-2}\sigma_1^{-1}\sigma_1^{-1} = \sigma_1^{-1}v_{i-2},$$

and thus $v_{i-2}$ commutes with $w$, which gives relation (4.3). If $1 \leq i, j \leq n - 3$ and $|i - j| \geq 2$, then the relator $\sigma_i\sigma_1^{-1}\sigma_1^{-1}\sigma_1^{-1}$ gives rise to the single relator $v_i^j v_i^{-1} v_j^{-1}$, while if $1 \leq i \leq n - 4$, the relator $\sigma_i\sigma_1^{-1}\sigma_1^{-1}\sigma_1^{-1}$ yields the single relator $v_i v_i+1 v_i = v_i+1 v_i v_i+1$; thus we obtain equations (4.1) and (4.2).

Now for $i = 1, \ldots, 2n - 2$, let $u_i = \sigma_i^{-1}\sigma_2\sigma_i^{-1}$. From the relator

$$\sigma_i^{j+1}\sigma_2\sigma_1\sigma_1^{-1}\sigma_2^{-1} = \sigma_i^{-1}\sigma_1^{-1} = \sigma_i^{-1} = (j-1),$$

we obtain the relators

$$u_{j+1} u_{j+2} u_{j+1}^{-1} \quad \text{if } j = 1, \ldots, 2n - 4,$$

$$u_{2n-3} w w_{1-1} u_{2n-2}^{-1} \quad \text{if } j = 2n - 3, \text{ and}$$

$$u_{2n-2} w w_{2n-2} u_{2n-2}^{-1} \quad \text{if } j = 2n - 2,$$

which yield respectively equations (4.9), (4.10) and (4.11).

If $2 \leq i \leq n - 3$, then the relator $\sigma_i^{-1}\sigma_2\sigma_2\sigma_3\sigma_2^{-1}\sigma_1^{-1}$ yields the relators

$$v_i u_{j+1} v_i^{-1} u_{j+1}^{-1} \quad \text{if } j = 1, \ldots, 2n - 3$$

which yield respectively equations (4.6), (4.7) and (4.8).

Finally,

$$\sigma_1 \sigma_2 \cdots \sigma_{n-2} \sigma_{n-1} \sigma_{n-2} \cdots \sigma_2 \sigma_1 = \sigma_1 \sigma_2 \sigma_1^{-2} \cdot \sigma_2^3 \sigma_3 \sigma_1^{-1} \cdot \cdots \cdot \sigma_1^{n-3} \sigma_{n-2}^{-1} \cdot \sigma_1 \sigma_2 \sigma_1^{-1} \sigma_3^2 \sigma_1^{-1} \cdot \sigma_1^{n-3} \sigma_{n-2}^{-1} \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_3^2 \sigma_1^{-1} \cdot \sigma_1\sigma_2 \sigma_1^{-1} \sigma_3^2 \sigma_1^{-1} \sigma_1^{-1},$$

and conjugating by $\sigma_i^{-1}$, we obtain relators

$$u_2 (v_1 \cdots v_n v_{n-1} v_{n-2} \cdots v_1) u_{2n-3} \quad \text{if } j = 1,$$

$$u_2 (v_1 \cdots v_n v_{n-1} v_{n-2} \cdots v_1) u_{2n-2} \quad \text{if } j = 2,$$

$$u_{j+1} (v_1 \cdots v_n v_{n-1} v_{n-2} \cdots v_1) w_{j+1} \quad \text{if } j = 3, \ldots, 2n - 3,$$

$$w_{j+1} (v_1 \cdots v_n v_{n-1} v_{n-2} \cdots v_1) u_{2n-4} \quad \text{if } j = 2n - 2.$$

This yields the remaining equations (4.12), (4.13), (4.14) and (4.15).

We now simplify somewhat the presentation of $\Gamma_2(B_n(S^2))$ given by Proposition 4.1. From equations (4.4) and (4.9), for $i = 1, 2$ we obtain the following
First suppose that we have equation (4.12). Now we may delete all but one of the surface relations; let us keep equation (4.12). This yields equation (4.15), which we delete from the list. It thus follows that we may write equations:

\begin{align*}
(4.4) & \quad v_j j = v_j j u_2 \quad \text{for all } j \geq 2, \\
(4.14) & \quad u_2 j = v_j j u_2^{-1} u_2 \quad \text{for all } j \geq 2.
\end{align*}

This allows us to eliminate equation (4.15) as follows. For all \( j \geq 2 \), we have

\[
v_j w u_1 w_j^{-1} = v_j w u_1^{-1} w_j^{-1} w^{-1} \quad \text{by equation (4.3)}
\]

\[
= w v_j w u_2^{-1} u_2^{-1} w^{-1} \quad \text{by equation (4.4)}
\]

\[
= w v_j w u_2^{-1} w^{-1} \quad \text{by equation (4.4)}
\]

\[
= v_j v_j w u_2^{-1} \quad \text{by equation (4.11)},
\]

and this is equivalent to equation (4.14), which we thus delete from the list of relations.

Suppose that for some \( 2 \leq i \leq 2n - 4 \), we have equations (4.4) and (4.4). We now show that they imply (4.4) + 1). For all \( j \geq 2 \), we have

\[
u_{i+1} v_j u_j^{-1} u_j^{-1} v_j^{-1} = u_{i+1}^{-1} u_j v_j u_j^{-1} u_j v_j^{-1} \quad \text{by equation (4.9)}
\]

\[
= u_j^{-1} v_j v_j v_j^{-1} \quad \text{by equation (4.1)}
\]

\[
= 1 \quad \text{by equation (4.4) - 1)}
\]

which yields equation (4.4) + 1). So we may successively delete equations (4.4) - 3), (4.4) - 4), ..., (4.4) from the list of relations.

Now we show that we may delete all but one of the surface relations (4.12) - (4.15). First suppose that we have equation (4.12). Now

\[
u_{2n-2} w u_3 = u_{2n-3} w u_1 u_3 \quad \text{by equation (4.10)}
\]

\[
= u_{2n-3} w u_2 \quad \text{by equation (4.1)}.
\]

This implies equation (4.13), which we delete from the list of relations.

Now suppose that we have equation (4.14) + 1) for some \( 5 \leq i \leq 2n - 2 \). Let us write \( A v_1 \cdots v_{i-2} v_{i-3} v_{i-4} v_1 \). Then \( w u_{i-2} u_{i+1} = A^{-1} \). So

\[
w u_{i-3} u_i = w u_{i-2} u_{i+1} \quad \text{by equation (4.9)}
\]

\[
= A^{-1} \quad \text{by above}.
\]

This yields equation (4.14), and so we may delete successively equations (4.14), ..., (4.14) - 3).

Now suppose that we have (4.15), so \( A v_{2n-4} w u_1 = 1 \). Then

\[
A v_{2n-4} w u_2 = A v_{2n-4} w u_1 w^{-1} \quad \text{by equations (4.9) and (4.10)}
\]

\[
= w(A v_{2n-4} w u_1) w^{-1} \quad \text{by equation (4.3)}
\]

\[
= 1 \quad \text{by above}.
\]

This implies equation (4.14) - 2), which we delete from the list of relations.

Finally, suppose that we have equation (4.12). Then

\[
A v_{2n-4} w u_1 = A v_{2n-3} v_{2n-2} w u_1 \quad \text{by equation (4.9) - 4)
\]

\[
= A v_{2n-3} w u_2 \quad \text{by equation (4.11)}
\]

\[
= 1 \quad \text{by above}.
\]

This yields equation (4.15), which we delete from the list. It thus follows that we may delete all but one of the surface relations; let us keep equation (4.12).
Summing up, we may thus delete relations (4.5), (4.4) for \( i = 3, \ldots, 2n - 3 \) and (4.13)–(4.15) from the presentation of \( \Gamma_2(B_n(S^2)) \) given by Proposition 4.1.

4.2. The derived subgroup of \( B_4(S^2) \). The aim of this section is to use Proposition 4.1 to derive the presentation of \( \Gamma_2(B_4(S^2)) \) given in Theorem 1.3(3), from which we were able to see that \( \Gamma_2(B_4(S^2)) \cong \mathbb{F}_2 \times \mathbb{Q}_8 \).

We first remark that in this case, the relations (4.1), (4.2), (4.4) and (4.5) do not exist. Further, from relations (4.9), we may obtain the following:

\[
\begin{align*}
&u_2 = u_3u_4^{-1} \quad u_1 = u_3u_4^{-1}u_3^{-1} \quad u_5 = u_3^{-1}u_4 \quad u_6 = u_4^{-1}u_3^{-1}u_4, \\
\end{align*}
\]

which we take to be definitions of \( u_1, u_2, u_5 \) and \( u_6 \), so we delete equation (4.11) from the list of relations. From equation (4.12), we see that

\[
w = u_4^{-1}u_3v_1^{-2}u_4u_3^{-1}.
\]

We conclude that \( \Gamma_2(B_4(S^2)) \) is generated by \( u_3, u_4 \) and \( v_1 \).

Let us return momentarily to the situation of the previous section. Before deleting all but one of the surface relations, we shall derive some other useful relations.

Consider the surface relations (4.12)–(4.15). From relations (4.12) and (4.14) (resp. (4.15) and (4.14)), it follows that \( u_5 \) (resp. \( u_4 \)) commutes with \( v_1 \). But these two equations are equivalent to the relations

\[
\begin{align*}
&(4.16) \quad u_3 \equiv v_1^2 \quad \text{and} \\
&(4.17) \quad u_4 \equiv v_1^2.
\end{align*}
\]

Further, equations (4.16) and (4.17) imply equation (4.14), and equations (4.12), (4.16) and (4.17) imply equation (4.14), so we replace equations (4.14) and (4.15) by equations (4.16) and (4.17).

As in Section 4.1, we can then delete equations (4.14b) and (4.15) from the list of relations, which becomes: (4.3), (4.4), (4.7), (4.8), (4.10) and (4.11). We now analyse these relations in further detail.

From equation (4.3) and the definition of \( w \), we see that \( v_1 \equiv u_4^{-1}u_3u_4u_3^{-1} \). Up to conjugacy, equation (4.9) may be written as follows:

\[
\begin{align*}
1 &= u_3v_1^{-1}u_3^{-1}u_3u_4u_3^{-1}v_1 = u_3v_1^{-1}u_3^{-1}u_4u_3^{-1}u_3u_4u_3^{-1}v_1 = u_3v_1^{-1}u_4v_1^{-1}u_4^{-1}v_1, \\
\end{align*}
\]

and hence we may replace equation (4.16) by

\[
(4.18) \quad u_3v_1u_3^{-1} = u_4v_1^{-1}u_4^{-1}v_1.
\]

Up to conjugacy, equation (4.6b) may be written:

\[
(4.19) \quad u_3^{-1}v_1u_3 = u_4^{-1}v_1u_4v_1^{-1}.
\]

By equations (4.18) and (4.19), the left-hand side of equation (4.6b) may be written:

\[
\begin{align*}
&u_3v_1u_3^{-1}u_4v_1^{-1}u_4^{-1}v_1^{-1} = u_3v_1^{-1}u_3^{-1}u_4u_3^{-1}v_1^{-1}, \\
\end{align*}
\]

so relation (4.6b) is automatically satisfied, and we thus delete it from the list.

Using the fact that \( v_1 \equiv u_4^{-1}u_3u_4^{-1} \), equation (4.10) may be written:

\[
\begin{align*}
1 &= u_4v_1u_4^{-1}u_3^{-1}u_4v_1^{-1}u_4^{-1}v_1 = u_4v_1u_3^{-1}u_3u_4u_3^{-1}u_4v_1^{-1}u_4^{-1}u_3v_1^{-1}, \\
&= u_4v_1u_3^{-1}u_4v_1^{-1}u_4^{-1}v_1^{-1},
\end{align*}
\]

and from this we obtain equation (4.19), using the fact that \( v_1^2 \) commutes with \( u_4 \). So we delete equation (4.6a) from the list.
We now consider equation (4.3). Using equations (4.18) and (4.19), we obtain
\[ 1 = u_4^{-1} u_3 u_4 u_3^{-1} v_1^{-1} u_3 u_4 u_4^{-1} v_1^{-1} = u_4^{-1} u_3 v_1 u_4 v_1^{-1} u_4^{-1} u_3^{-1} u_4 v_1^{-1} = u_4^{-1} u_3 v_1 u_4 v_1^{-1} u_4^{-1} v_1^{-1} u_4 = u_4^{-1} u_3 v_1 u_3^{-1} v_1^{-1} u_4, \]
which up to conjugacy, and using the fact that \( v_1^2 \) commutes with \( u_3 \), yields
\[ (4.20) \quad u_3^{-2} v_1^{-1} u_3 v_1^{-1} u_3 v_1^{-1} \cdot v_1^4 = 1. \]
We replace equation (4.20) by this relation.

From equations (4.16) and (4.17), the left-hand side of equations (4.10) and (4.11) collapses, and so we delete these equations from the list.

After immediate cancellations, equation (4.8) becomes
\[ 1 = u_4^{-1} u_3 v_1 u_4^{-1} u_3^{-1} u_4 v_1^{-1} = u_4^{-1} u_3 v_1 u_4^{-1} v_1^{-1} v_1 u_3 v_1^{-1} u_4 v_1^{-1} u_4 v_1^{-2} = u_4^{-1} u_3 v_1 u_3^{-1} v_1 u_3 u_3 v_1^{-1} u_3^{-1} u_4 v_1^{-2}, \]
which up to conjugacy and inversion yields equation (4.20). So we delete equation (4.20) from the list.

After immediate cancellations, the left-hand side of equation (4.7) becomes
\[ u_3^{-1} u_4 v_1 u_4^{-1} u_3 u_4^{-1} u_3^{-1} u_4 v_1^{-1} = u_3 u_4 v_1^{-1} = u_3^{-1} u_4 v_1 u_4^{-1} v_1^{-1} u_3 v_1^{-1} = u_3^{-1} u_3 v_1 u_3^{-1} u_3 v_1^{-1} = 1, \]
using the fact that \( v_1 = u_4^{-1} u_3 v_1 u_3^{-1} \) and applying equations (4.18) and (4.17). So we delete equation (4.7) from the list.

We are thus left with relations (4.16), (4.17), (4.18), (4.19) and (4.20). We now multiply together equations (4.18) and (4.19). The product of the left-hand sides, by equation (4.20), is given by
\[ u_3 v_1 u_3^{-2} v_1 u_3 = v_1, \]
while by equations (4.16), (4.17), (4.18), (4.19) and (4.20), the product of the right-hand sides is given by
\[ u_4 v_1^{-1} u_4^{-1} v_1 u_4 v_1^{-1} = u_4 v_1^{-1} u_4^{-1} v_1 u_4 v_1^{-1} = v_1^{-1} v_1 u_4 v_1^{-1} u_4^{-1} u_3^{-1} v_1^{-1} u_3 v_1^{-1} = v_1^{-1} u_4 v_1^{-1} u_3^{-1} v_1^{-1} u_3 v_1 = v_1^{-1} u_3 v_1^{-1} u_3^{-2} v_1^{-1} u_3 v_1 = v_1^{-3}. \]
From these two equations, we conclude that
\[ (4.21) \quad v_1^4 = 1, \]
and so equation (4.20) becomes
\[ (4.22) \quad u_3^{-2} v_1^{-1} u_3 v_1^{-1} u_3 v_1^{-1} = 1. \]

The list of relations now becomes (4.16), (4.17), (4.21), (4.22), (4.18) and (4.19). We may rewrite the corresponding presentation as follows:

**Proposition 4.3.** The following constitutes a presentation of the group \( \Gamma_2(B_3(S^2)) \):

**generators:** \( g_1, g_2, g_3 \), where in terms of the usual generators of \( B_4(S^2) \), \( g_1 = u_3 = \sigma_1^2 \sigma_2 \sigma_1^{-3} \), \( g_2 = u_4 = \sigma_1^3 \sigma_2 \sigma_1^{-4} \) and \( g_3 = v_1 = \sigma_3 \sigma_1^{-1} \).
relations:
\[ g_3^4 = 1, \]
\[ g_3^2 = g_1, \]
\[ g_3^3 = g_2, \]
\[ g_3 = g_2 g_1. \]

Proof. Rewriting \( u_3, u_4 \) and \( v_1 \) in terms of the \( g_i \), we obtain directly the first three and the last of the given relations. As for the fourth and fifth relations, we obtain respectively
\[ g_3 g_2 g_1 g_3^{-1} g_1^{-1} g_2^{-1} = g_2^{-1} g_1^{-1} g_2^{-1} g_1 g_2 g_3^{-1} = 1, \]
\[ g_1^{-2} g_3^{-1} g_1^{-1} g_3^{-1} g_1 g_2 g_3^{-1} = 1. \]

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