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JOURNAL OF MATHEMATICAL PHYSICS, v.51, n.10, 2010
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Free-fall in a uniform gravitational field in noncommutative quantum mechanics

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(Received 14 April 2010; accepted 25 June 2010; published online 12 October 2010)

We study the free-fall of a quantum particle in the context of noncommutative quantum mechanics (NCQM). Assuming noncommutativity of the canonical type between the coordinates of a two-dimensional configuration space, we consider a neutral particle trapped in a gravitational well and exactly solve the energy eigenvalue problem. By resorting to experimental data from the GRANIT experiment, in which the first energy levels of freely falling quantum ultracold neutrons were determined, we impose an upper-bound on the noncommutativity parameter. We also investigate the time of flight of a quantum particle moving in a uniform gravitational field in NCQM. This is related to the weak equivalence principle. As we consider stationary, energy eigenstates, i.e., delocalized states, the time of flight must be measured by a quantum clock, suitably coupled to the particle. By considering the clock as a small perturbation, we solve the (stationary) scattering problem associated and show that the time of flight is equal to the classical result, when the measurement is made far from the turning point. This result is interpreted as an extension of the equivalence principle to the realm of NCQM. © 2010 American Institute of Physics. [doi:10.1063/1.3466812]

I. INTRODUCTION

The idea that space-time would have a noncommutative structure was proposed by Heisenberg and others,\textsuperscript{1} already at the very beginning of quantum field theory (QFT), so that an effective ultraviolet cutoff could be introduced at very small length scales, in an attempt to get rid of divergences. It was only in 1947, with Snyder,\textsuperscript{2} that this idea of space-time noncommutativity was formalized (for a historical introduction, see Refs. 3 and 4). But due to the success achieved by the renormalization program of QFTs, relatively low interest remained on the subject. In the late 1990s some results coming from string theory have suggested that space-time may display a noncommutative structure,\textsuperscript{5} thus starting a great revival on the study of QFTs based on noncommuting space-time coordinates (for reviews, see Refs. 3, 6, and 7). Nevertheless, we remark that the issue of space-time noncommutativity was considered in an earlier work by Doplicher et al.,\textsuperscript{8} who constructed a unitary QFT based on a noncommutative space-time, motivated by the issue of the description of the quantum nature of the space-time.

As quantum mechanics can be considered the one-particle sector of quantum field theory, it is interesting to study the quantum mechanics defined on noncommutative spaces. Lately, this noncommutative quantum mechanics (NCQM) has been increasingly studied (see, for instance, Refs 9–11) and the effects of noncommutativity that might be experimentally detectable have been investigated.\textsuperscript{12,13} There have been also studies of many-particle systems\textsuperscript{14} and an approach to

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NCQM that is directly related to the ideas developed in Ref. 8 has been developed in Ref. 15, by means of an algebraic setup in which the time is a noncommutative coordinate as well as the space coordinates, with interesting ensuing consequences (see Ref. 16).

In this paper we are concerned about the physics of a quantum particle in a uniform gravitational field. In the first part of the paper we study the problem known as the gravitational quantum well. This can be obtained by the Earth’s gravitational field and a perfectly reflecting mirror at the bottom. In the context of ordinary quantum mechanics, this well-known problem has been thoroughly studied in textbooks and pedagogic articles (see, for instance, Ref. 17). But this simple theoretical investigation gained a lot in importance, as recently such a quantum well was experimentally realized and the first two quantum states of neutrons moving in the Earth’s gravitational well were identified. This was achieved through the GRANIT experiment, performed by Nesvizhevsky et al.18,19 In this paper we study the noncommutative version of this problem, by means of the solution of the noncommutative Schrödinger equation. As it will be shown, the experimental results obtained in Refs. 18 and 19 allow us to find an upper-bound on the parameter of the spatial noncommutativity.

The first study to treat the noncommutative gravitational quantum well was done by Bertolami et al.,20 but in a noncommutative model different from the one we will consider in this paper. We remark that we have assumed only spatial noncommutativity, in contrast to other works in the literature, which considered noncommutativity of both configuration and momentum spaces (see Refs. 20–24) or time-space noncommutativity (see Ref. 22). References 20–23 treated the noncommutative gravitational quantum well and used data from the GRANIT experiment18,19 to find upper-bounds on the value of the momentum-momentum noncommutativity parameter, while in Ref. 22, an upper-bound on the time-space component of the noncommutative matrix was found, by means of second quantization techniques. Nonetheless, differently from Refs. 20–23, we have found the shifts in the energy levels due to spatial noncommutativity only. This was done by studying the adjointness of the Hamiltonian operator associated with the problem, what led us to determine the self-adjoint extensions of it. We have treated the problem without resorting to any perturbative approach.

Although studies of noncommutative corrections to general relativity have been addressed in the literature (see, e.g., Ref. 25), in our work we study the quantum mechanics of a particle with noncommutative spatial degrees of freedom and subject to a Newtonian gravitational field. In fact, since in the GRANIT experiment one considers the gravitational field near the Earth’s surface, where the Newtonian gravity is an excellent approximation, one can treat quantum mechanics with spatial noncommutative degrees of freedom as a fairly well approximation, without evoking the possible noncommutative structure that gravity might display. As we are not considering gravity as being described by general relativity, we did not care about noncommutative corrections to the Hilbert–Einstein action. Furthermore, the spatial noncommuting coordinates that we are considering are not the coordinates of the underlying space-time, but rather of the configuration space. The assumption of noncommutativity of the configuration space does not necessarily imply a noncommutative space-time. One can, for instance, study a noncommutative scalar field coupled to gravity in a curved space-time, without taking into account modifications of gravity due to noncommutativity (see, e.g., Ref. 26). We stress that we are studying a nonrelativistic quantum mechanics and that the noncommutativity in this case is conceptually different from that of relativistic quantum mechanics or quantum field theory, where the space-time itself is to be considered noncommutative. As remarked, for instance, in the Refs 27 and 28, in NCQM the noncommutativity modifies only the algebra of the basic observables of the theory and does not introduce modifications in the structure of the underlying space-time (see also Ref. 29 and references therein).

In the second part of this paper we consider the issue of the equivalence principle in the context of NCQM. In ordinary quantum mechanics, this is an interesting and subtle question (see, e.g., Ref. 30). We address the issue of the (weak) equivalence principle in NCQM motivated by the study due to Davies in the case of ordinary quantum mechanics.31 Although (i) classically the equivalence principle has a local character, and (ii) in a uniform gravitational field the Schrödinger equation leads to mass-dependent results, one can still say that the equivalence principle is real-
ized at the quantum level if the motion of a quantum particle is considered to be described by a wave packet. This is a consequence of the Ehrenfest’s theorem (see discussion in Ref. 31). Nevertheless, Davies asked about the validity of the equivalence principle in the case of delocalized quantum states, such as energy eigenstates. These do not have classical counterparts associated with a localized particle with a well-defined trajectory. In order to analyze this, Davies considered a variant of the (gedanken) experiment of Galileo at the leaning Tower of Pisa, with particles of different masses that would be vertically projected up in a uniform gravitational field. For classical particles, in the neighborhood of the Earth, for instance, the out-and-back time is twice that spent in climbing, but for quantum particles, there is a nonvanishing probability that they tunnel into the classically forbidden region. This special feature of the quantum theory might then give rise to a departure from the classical turnaround time, since a delay might ensue. In ordinary quantum mechanics, when one considers stationary, energy eigenstates, Davies showed that, when the measurement is made far from the classical turning point, the time of flight of a quantum particle is identical to the classical result.31 In this sense the (weak) equivalence principle is preserved in quantum mechanics. In order to measure the time of flight associated with delocalized states, Davies applied a simple model of quantum clock, due to Peres.32 This clock runs only when the particle travels within a given region of interest and does not measure absolute instants of time, but just the time difference.

It is in the sense mentioned above that in this paper we show that the equivalence principle can be extended to the NCQM in the case of stationary, energy eigenstates. Nevertheless, we remark that we achieve this result by explicitly solving the (stationary) scattering problem of a quantum particle in the presence of a uniform gravitational field. Our approach to the problem does not closely follow that of Davies, but rather we closely follow the original approach by Peres32 and explicitly consider that the quantum clock is coupled to the particle. Then, we will show that, in the case of small interaction between the particle and the clock, and when the measurement is made far from the turning point, the time of flight will be given by the phase shift of the wave-function and this shift correctly codifies the time as measured by the quantum clock.

This paper is organized as follows. In Sec. II, we review the basic features of the formalism of NCQM. In Sec. III, we study the noncommutative gravitational quantum well. In Sec. IV, we establish an upper-bound on the noncommutativity parameter. In Sec. V, we review the Peres quantum clock model. In Sec. VI, we apply it to the investigation of the time of flight of a particle subjected to a uniform gravitational field in NCQM. Finally, in Sec. VII we make concluding remarks.

II. NONCOMMUTATIVE QUANTUM MECHANICS

Quantum mechanics inspired the idea of noncommutative coordinates, which can be introduced by (see, for instance, Ref. 3)

\[ [\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \]

where \( \hat{x}_\mu \) are the coordinate operators (the noncommutative analogs of the ordinary coordinate functions), \( \mathbb{1} \) is the unit element of the noncommutative algebra, and \( \theta_{\mu\nu} \) is the antisymmetric, real-valued \( d \times d \) matrix describing the coordinate noncommutativity (\( d \) is the space-time dimension). The uncertainty relations \( \Delta x_\mu \Delta x_\nu \geq |\theta_{\mu\nu}|/2 \), which are compatible with commutation relations (1) allows one to think of \( |\theta_{\mu\nu}| \) as the characteristic scale of length involved in the uncertainty on the simultaneous measurement of the coordinates \( \hat{x}_\mu \) and \( \hat{x}_\nu \). In this paper we will set \( \theta_{0j} = 0 \), since we are not interested in dealing with the possibility of violation of causality33 and unitarity.34 The three-dimensional noncommutative space generated by \( \hat{x}_i \) and \( \mathbb{1} \) will be denoted by \( \mathbb{R}^3_\theta \).

In order to implement quantum physics we need to introduce the observables and the evolution equation. The position observables can be introduced by means of the left-action of \( \mathbb{R}^3_\theta \) on itself, while the momenta \( \hat{p}_i \) commute with each other and are canonically conjugated to the position operators. The phase-space commutation relations then read
\[
\{\hat{x}_i, \hat{\xi}_j\} = i\theta_{ij}, \quad [% \hat{p}_j, \hat{x}_i\} = 0, \quad [% \hat{p}_j, \hat{\xi}_i\} = -i\hbar \delta_{ij}.
\] (2)

We remark that our phase-space commutation relations are distinct from the ones used in Refs. 20–23, where not only the coordinates but also the momenta are noncommutative. According to Ref. 20, such a phase-space noncommutativity leads to a modification of the Planck’s constant, whereas Ref. 21 argues that this is not needed. Nevertheless, according to Ref. 35, those approaches are, in fact, physically equivalent, differing only in the manner one defines the noncommutative parameters. On the other hand, according to Refs. 20 and 36, the type of phase-space algebra studied in Refs. 20–23 can be led to the canonical form by means of an appropriate change of variables. Moreover, this change of variables is not unique, as showed in Ref. 37.

The noncommutative phase-space operators in Eq. (2) can be written in terms of the ordinary position and momentum operators by means of the following (noncanonical) transformation (see Refs. 27, 28, and 38, for example):

\[
q_i = \hat{x}_i + \frac{\theta_{ij}}{2\hbar} \hat{p}_j, \quad p_i = \hat{p}_i,
\] (3)

where \(q_i\) and \(p_i\) satisfy \([q_i, p_j] = i\hbar \delta_{ij}\). Besides that, we can represent the algebra (2) of observables of NCQM on the same representation space as the Heisenberg algebra (see, e.g., Refs. 27 and 28), what means that the set of states in NCQM is the same as in ordinary quantum mechanics. This is interpreted as a manifestation of the fact that the noncommutativity introduced by Eq. (1) has no observable consequences at the level of kinematics.66 The nontrivial effects coming from the space-space noncommutativity are due to dynamical considerations.

Before we state the basic dynamical equation, we note that if we considered a charged quantum neutral particle moving under the action of a Newtonian gravitational potential is not related to a gauge theory, we cannot apply the gauge principle to study that interaction. Hence, we will follow a quite standard approach that has been extensively used in the literature on NCQM, which consists in starting with the noncommutative Schrödinger equation

\[
-\frac{\hbar^2}{2m} \nabla^2 \Psi + V \Psi = i\hbar \frac{\partial \Psi}{\partial t},
\] (4)

where \(\star\) denotes the so-called Moyal product, defined by

\[
(\psi \star \phi)(x_1, x_2, x_3) = \psi(x_1, x_2, x_3)e^{(i/2)\delta\theta_{ij}}\phi(x_1, x_2, x_3),
\] (5)

which plays a major role, as it gives the action of the interaction Hamiltonian on wave-functions. We would like to emphasize that this approach has been applied by several authors to a variety of problems (see, for example, Refs. 9, 10, 27, and 28).

An equivalent way to introduce the evolution equation (see, for example, Refs. 9, 27, and 28) is to start with the ordinary Schrödinger equation and then substitute the ordinary (commutative) coordinates by the noncommutative ones, by making use of the inverse of Eq. (3). As a consequence the interaction Hamiltonian gets modified by a shift (sometimes called Bopp shift), that is,

\[
-\frac{\hbar^2}{2m} \nabla^2 \Psi + V \left(x_i + \frac{i\theta_{ij}}{2} \frac{\partial}{\partial x_j}\right) \Psi = i\hbar \frac{\partial \Psi}{\partial t}.
\] (6)

By making use of (5) one can show that Eq. (6) is equivalent to Eq. (4) (see, e.g., Ref. 27).

The energy eigenvalue equation of NCQM can be obtained by means of the usual stationary problem ansatz, \(\Psi(x, y, z, t) = \psi(x, y, z)e^{-iEt/\hbar}\). The corresponding time-independent equation thus reads
\[-\frac{\hbar^2}{2m}\nabla^2\psi + V\star\psi = E\psi, \tag{7}\]

where \(E\) is the energy of the particle.

Equations (4) and (7) are the starting point of many calculations in NCQM that have been considered in the literature. They can be viewed as the Schrödinger equation of ordinary quantum mechanics, but with the interaction term modified by a contribution coming from the noncommutativity of the configuration space. That means that the problems in NCQM can be approached in the same way as we would do in ordinary quantum mechanics, the only difference being the replacement of the ordinary potential energy by its deformed \((\theta_{ij}\text{-dependent})\) version. In this sense, the NCQM corresponds to a true deformation of ordinary quantum mechanics, and as \(\theta_{ij}\) goes to zero we recover the ordinary results. Some phenomenological consequences of this deformation will be addressed below, when we consider the two-dimensional motion of a quantum neutral particle at the presence of an external gravitational potential.

III. THE NONCOMMUTATIVE GRAVITATIONAL QUANTUM WELL

In this section we consider the noncommutative version of the gravitational quantum well and determine the energy spectrum of a particle trapped in it.

Usually, the correct definition of an operator is not addressed in most applications in physics, but rather an operator is defined by means of its law of action (the so-called formal operator) only, with no mention about its domain of definition. Operators having the same formal expression but acting in different domains can lead to different physics. This is an important question especially in quantum theory.\(^40\) Bearing this in mind, in order to determine the spectrum of the noncommutative gravitational well, we will carefully examine the domain of the differential operators we have to handle with. This is closed related to the self-adjointness of the operator and will ultimately lead us to consider its self-adjoint extensions [an operator \(S\) is an extension of an operator \(T\) if \(D(T) \subset D(S)\) and \(S\phi = T\phi\), for all \(\phi \in D(T)\), where \(D(T)\) and \(D(S)\) are the domains of definition of \(T\) and \(S\), respectively]. This is a crucial step to correctly solve the eigenvalue problem associated with the quantum well.

A. The specification of the model

Let \(\tilde{g}\) be a uniform gravitational field and let the \(Oy\)-axis be oriented in the opposite direction of \(\tilde{g}\), i.e., \(\tilde{g} = -\tilde{g}_y\). At the boundary \(y = 0\) the particle encounters a perfectly reflecting mirror, which prevents it to get into the negative portion of the \(Oy\) axis. We thus have

\[
V(x,y,z) = \begin{cases} 
mg y, & y > 0, \quad \forall x, z \\
\infty, & y = 0, \quad \forall x, z.
\end{cases} \tag{8}
\]

Since the particle configuration space is \(Q = \mathbb{R}^2 \times (0, \infty) = \mathbb{R}^2 \times \mathbb{R}^*_+\), we have to be cautious because the Moyal product is not well-defined on manifolds with boundaries,\(^41\) a fact that can be traced back to the nonlocality of Eq. (5). In order to avoid this problem, we consider the \(\theta_{ij}\) as sufficiently small parameters, so that we can truncate the series of the Moyal product at some suitable power of \(\theta_{ij}\). For example, the first order \(\theta_{ij}\)-dependent correction to the ordinary potential is

\[
V\star\psi \approx V\psi + \frac{i}{2} \theta_{ij} \frac{\partial V}{\partial x_i} \frac{\partial}{\partial x_j} \psi. \tag{9}
\]

The most important feature of this approximation is its local character, which allows one to work in the realm of the upper half-space, without having to worry about the issues coming from the nonlocality of the Moyal product. Thus, Eq. (9) leads to the following time-independent Schrödinger equation:
\[
H\psi = -\frac{\hbar^2}{2m}\nabla^2 \psi + mg_y \psi + \frac{im \theta_{21}}{2} \frac{\partial \psi}{\partial x} + \frac{im \theta_{23}}{2} \frac{\partial \psi}{\partial z} = E\psi. \quad (10)
\]

It is interesting to notice that the direct application of Eq. (5) to the case of the potential in Eq. (8) would lead to Eq. (10) too: in the case of a simple linear potential, the first order approximation considered in Eq. (9) is, in fact, exact. Nevertheless, in the presence of boundaries, the Moyal product would lead to inconsistencies when applied to the calculation of nonlinear quantities, such as transition amplitudes, provided that the wave-function is nonpolynomial. Thus, we will consider approximation (9), as we said.

Although the Hamiltonian in Eq. (10) acts on wave-functions of the form \(\psi(x, y, z)\), the noncommutative gravitational quantum well is a genuine two-dimensional problem, just like its commutative counterpart. Indeed, it can be shown (see in the following) that Hamiltonian (10) is invariant under passive rotations around the axis of the gravitational field, which, in our case, was taken to coincide with the \(Oy\)-axis of the coordinate system. Because of this symmetry, there is no loss of generality if we consider, for simplicity, that the initial momentum of the particle has zero component along one of the horizontal axes. In order to verify this we recall that the effect of a rotation \(R(\beta)\) of the observer around the \(Oy\)-axis is given by (in our convention a positive rotation angle \(\beta\) corresponds to a counterclockwise rotation of the observer around the \(Oy\)-axis)

\[
\hat{x}_1' = \cos(\beta)\hat{x}_1 - \sin(\beta)\hat{x}_3, \quad \hat{x}_2' = \hat{x}_2, \quad \hat{x}_3' = -\sin(\beta)\hat{x}_1 + \cos(\beta)\hat{x}_3,
\]

while the matrix \(\theta_{ij}\) transforms as a tensor, that is, \(\theta_{ij}' = R_{ik}(\beta)R_{jl}(\beta)\theta_{kl} = R_{ik}(\beta)\theta_{ij}R^{-1}_{ij}(\beta)\). By taking into account the momentum operator transformation rule under a passive rotation around the \(Oy\)-axis,

\[
p_{l}' = e^{-i(\hbar/\mu_l)}p_{l}e^{i(\hbar/\mu_l)}, \Rightarrow \begin{cases}
    p_{x}' = \cos(\beta)p_{x} - \sin(\beta)p_{z} \\
    p_{y}' = p_{y} \\
    p_{z}' = \sin(\beta)p_{x} + \cos(\beta)p_{z},
\end{cases}
\]

one can directly verify that the transformed Hamiltonian \(H'\) is such that \(H' = H\), as stated.

From the above, we can assume, for simplicity, that the motion takes place in a plane parallel to the \(xy\)-plane, without any loss of generality. We will make use of this fact in Sec. III B.

Finally, we remark that we are simply studying the quantum mechanics of a particle with noncommutative spatial degrees of freedom and subject to a Newtonian gravitational field. Of course, in principle, one might consider even noncommutative corrections to general relativity, but these are expected to be of order \(\hbar^2\), whereas NCQM generally leads to leading corrections of first order in \(\theta_{ij}\). For example, in Ref. 25 a noncommutative version of the coupling of classical gravity to a classical test particle was studied and the leading-order noncommutative correction to the Newtonian potential, in the linearized approximation, was shown to have the form \(V' = V_{\text{Newton}} + O(\theta_{ij}^2)\). This fact is specially interesting for us, as it indicates that the noncommutative effects coming from the deformation of general relativity and the deformation of the quantum dynamics that we are considering in the present paper do not mix each other.

**B. Domains, self-adjointness, and Hamiltonian spectrum**

From now on we assume that the motion of the particle takes place in a \(z = \text{constant}\) plane, so that \(\psi\) is a function of the \(x\) and \(y\) variables solely. The only component of the noncommutativity that matters to our purposes is \(\theta_{12}\). Hence, in what follows we will simplify the notation, referring to \(\theta_{12}\) simply as \(\theta\). Thus, Eq. (10) reduces to

\[
H\psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} - \frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial y^2} + mg_y \psi - \frac{mg_i\theta \partial \psi}{2} = E\psi. \quad (13)
\]

As it stands, the Schrödinger equation, Eq. (13), is purely formal. In order to properly address the energy eigenvalue problem, one must define the domain of the Hamiltonian operator \(H\).
Since the formal expression of $H$ involves differentiation in the strong sense, that is, in the sense of advanced calculus, it is not well-defined on the whole Hilbert space $\mathcal{H}$. In such a case we have to restrict the action of $H$ to a dense subset, $D(H) \subset \mathcal{H}$, which we shall call the domain of $H$ (operators defined on dense domains are called densely defined operators). The restriction to dense subsets guarantees the existence of the adjoint operator, a necessary condition for one to be able to talk about the self-adjointness of an operator. We point out that the operators which can be defined on the whole Hilbert space and satisfy $(T\phi, \psi) = (\phi, T\psi), \forall \phi, \psi \in D(T)$ (i.e., that are Hermitian) are necessarily bounded. On the other hand, an unbounded operator cannot be defined as a self-adjoint operator for all vectors of the Hilbert space. It turns out that Hamiltonian (13) is an unbounded operator, since $D(H)$ is a proper subset of $\mathcal{H}$.

As the configuration space of a particle trapped inside the gravitational quantum well is $Q = R \times R^*_x$, its Hilbert space of states is $\mathcal{H} = l^2(Q) = l^2(R) \otimes l^2(R^*_x)$. The domain of the Hamiltonian is $D(H) = C_0^\infty(R) \otimes C_0^\infty(R^*_x)$, where $C_0^\infty(R)$ denotes the space of functions $\phi: R \rightarrow C$, such that $\phi$ is infinitely differentiable and have compact support, which means that the set of points where $\phi$ is not zero is a bounded and closed subset of $R$ [the space $C_0^\infty(R^*_x)$ is defined in an analogous way]. Of course, the choice of such a $D(H)$ was motivated by the fact that $H$ is naturally split into two independent parts, i.e., $H = H_x \otimes 1_{D(H_y)} + 1_{D(H_x)} \otimes H_y$, where

$$H_s = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \frac{mg \theta}{2} \frac{\partial}{\partial x}, \quad H_y = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + mgy,$$

(14)

$D(H_s) = C_0^\infty(R)$ and $D(H_y) = C_0^\infty(R^*_x)$.

We now investigate the self-adjointness of the Hamiltonian in its domain. By definition, an operator is self-adjoint if it is equal to its adjoint. That means that $D(T) = D(T^*)$ and $(T\phi, \psi) = (\phi, T\psi)$ for all $\phi, \psi \in D(T)$. Therefore, the self-adjointness of a densely defined operator is equivalent to Hermiticity plus the equality $D(T) = D(T^*)$. Only when an operator is bounded is that Hermiticity implies self-adjointness. In order to verify the Hermiticity of $H$, we perform integration by parts and make use of the reality of $V(y)$ and the fact that the functions belonging to $D(H)$ vanish at the boundaries (they have compact support). Even though the Hamiltonian is Hermitian, it turns out that it is not self-adjoint, since the domain of $H^*$ is larger than the domain of $H$. Thus, $H^*$ is a nontrivial extension of $H$, i.e., $D(H) \subset D(H^*)$ and $H^* \phi = H \phi$ for all $\phi \in D(H)$. In what follows we verify these statements.

The structure of the Hamiltonian (including the product structure of its domain) means that the action of $H$ on elements of $D(H)$ can be found once we know the actions of $H_x$ and $H_y$ into their respective domains. Besides, the adjoint of the Hamiltonian can be expressed as a linear combination of $H_x^*$ and $H_y^*$, each one of which can be found separately. We consider $H_x$ at first. It is a linear combination of the differential operators $p_x := -i\partial_x$ and $h_x := -\partial_x^2$, both defined on $C_0^\infty(R)$. By definition, if $\psi \in D(p_x^*)$ then $(p_x \phi, \psi) = (\phi, p_x^* \psi)$ for all $\phi \in C_0^\infty(R)$. Therefore,

$$\int_R dx \overline{\phi(x)} \psi(x) = -\int_R dx \overline{\phi(x)} (ip_x^* \psi)(x)$$

(15)

for all $\phi \in C_0^\infty(R)$. According to the definition of weak derivative, it follows that $(p_x^* \psi)(x) = -i\psi'(x)$, where $\psi'(x)$ is the weak derivative of $\psi \in D(p_x^*)$ with respect to $x$. Besides, since the range of $p_x$ is a subset of $L^2(R)$, it results that

$$D(p_x^*) = \{ \psi \in L^2(R) : \psi \in L^2(R) \}.$$

(16)

Thus, we may write $D(p_x) = C_0^\infty(R) \subset D(p_x^*)$, which means that the domain of $p_x$ is a proper subset of the domain of its adjoint.

Regarding $h_x^*$, the defining equation is $(h_x \phi, \psi) = (\phi, h_x^* \psi)$, which can be written as
\[
\int_{\mathbb{R}} dx \bar{\phi}(x) \psi(x) = \int_{\mathbb{R}} dx \bar{\phi}(x)(-h^*_\alpha \psi)(x)
\]  

(17)

for all \( \phi \in C_c^0(\mathbb{R}) \). It results that \( (h^*_\alpha \psi)(x) = -\psi''(x) \), where \( \psi''(x) \) is the second weak derivative of \( \psi \in D(h^*_\alpha) \) with respect to \( x \). Since the range of \( h^*_\alpha \) is a subset of \( L^2(\mathbb{R}) \), it follows that \( \psi'' \) is square integrable. Thus, we can write

\[
D(h^*_\alpha) = \{ \psi \in L^2(\mathbb{R}) : \psi'' \in L^2(\mathbb{R}) \},
\]

(18)

so that \( D(h_\alpha) = C_c^0(\mathbb{R}) \subset D(h^*_\alpha) \), which means that the domain of \( h_\alpha \) is a proper subset of the domain of its adjoint.

According to (16) and (18), neither \( p_\alpha \) nor \( h_\alpha \) are self-adjoint in their respective domains, implying that \( H_\alpha \) is not self-adjoint too. Nevertheless, what is important to notice is that both \( p^*_\alpha \) and \( h^*_\alpha \) are self-adjoint. Let us take \( p^*_\alpha \), for example. If \( \tilde{\psi}(p) = (\mathcal{F}\psi)(p) \) denotes the Fourier transform of \( \psi \in D(h^*_\alpha) \), we may write the following familiar results, which can be demonstrated by means of the usual Fourier transform techniques:\(^{40,45}\)

\[
(\mathcal{F}p^*_\alpha \psi)(p) = p \tilde{\psi}(p), \quad (\mathcal{F}p^*_\alpha \mathcal{F}^{-1}\tilde{\psi})(p) = p \tilde{\psi}(p).
\]

(19)

We know from Eq. (16) that \( p^*_\alpha \psi \) is square integrable. Since the Fourier transform is a unitary map from \( L^2(\mathbb{R}) \) into itself (see Ref. 40), it follows that \( p \tilde{\psi}(p) \) is square integrable too. Moreover, Eq. (19) implies that the operator \( p^*_\alpha \) is unitarily equivalent to the operator \( M_p \) of multiplication by \( p \). Since the multiplication operator is self-adjoint in \( D(M_p) = \{ \psi \in L^2(\mathbb{R}) : M_p \psi \in L^2(\mathbb{R}) \} \), then \( p^*_\alpha \) is self-adjoint too. We may write \( p^*_\alpha = \mathcal{F}^{-1}M_p \mathcal{F} \). A completely analogous result is valid for \( h^*_\alpha \), with \( D(p^*_\alpha) \), \( M_p \), and \( D(M_p) \) replaced by \( D(h^*_\alpha) \), \( M_{\psi} \), and \( D(M_{\psi}) \), respectively. It results that by an appropriate choice of its domain, \( D(H^*_\alpha) \), the formal differential operator

\[
H^*_\alpha = \frac{\hbar^2}{2m^*} + \frac{mg \theta}{2} p^*_\alpha
\]

(20)

can be made self-adjoint. Let \( D_0 = D(h^*_\alpha) \cap D(p^*_\alpha) \). The operator \( H^*_\alpha \) is Hermitian in \( D_0 \), but it turns out that this domain is unnecessarily restrictive \( (D_0 \subset D(H^*_\alpha)) \). In order to characterize \( D(H^*_\alpha) \), we make use of some results from the theory of partial differential equations, notably the Sobolev embedding theorem.\(^{45}\)

From the above results we know that if \( \psi \in D_0 \) then \( \psi, \psi', \psi'' \in L^2(\mathbb{R}) \), which means that \( \psi \) belongs to the Sobolev space \( W^{1,2} = H^2 \), the space of the square integrable functions with square integrable weak derivatives of first and second orders. Therefore, according to the Sobolev embedding theorem,\(^{45}\) \( \psi \) is a continuously differentiable function, that is, \( \psi \in C^1(\mathbb{R}) \).

Now we make use of the characterization of Sobolev spaces in terms of the absolute continuity. A function \( f: \mathbb{R} \to \mathbb{C} \) is absolutely continuous in \( I = [a, b] \) if and only if there is an integrable function \( g:I \to \mathbb{C} \), such that \( f(x) = f(a) + \int_a^x dg(\xi), \quad \forall x \in [a, b] \). If \( f \) is absolutely continuous in every \( [a, b] \subset \mathbb{R} \) then we say that \( f \) is locally absolutely continuous in \( \mathbb{R} \) and write \( \psi \in AC(\mathbb{R}) \). This characterization of Sobolev spaces involves the notion of “absolute continuity on lines,” but in the particular case of functions of one real variable, we only need the notion of local absolute continuity (see Ref. 46, for details). In order to be allowed to use this result we suppose that \( \psi'' = (\psi')' \). Then it follows that if \( \psi \in D_0 \) then \( \psi' \in W^{1,2} \), the space of square integrable functions with square integrable first order weak derivative. According to Ref. 46 we have \( W^{1,2} = AC(\mathbb{R}) \cap L^2(\mathbb{R}) \), so that \( \psi' \) is locally absolutely continuous.

The above discussion reveals the structure of the domain of self-adjointness of \( H^*_\alpha \). We notice that it is perfectly possible to drop the much more restrictive requirements of square-integrability of \( \psi' \) and \( \psi'' \), since only the range of \( H^*_\alpha \) must be square integrable, that is, \( H^*_\alpha \psi \in L^2(\mathbb{R}) \). It results that the domain of self-adjointness of \( H^*_\alpha \) can be written as
\[
D(H_s^y) = \left\{ \psi \in L^2(\mathbb{R}) \cap C^1(\mathbb{R}); \psi' \in AC(\mathbb{R}), \quad -\frac{\hbar^2}{2m} \psi'' - \frac{mg\theta}{2} \psi' \in L^2(\mathbb{R}) \right\}.
\]

For a good introduction on the importance of Sobolev spaces in quantum physics, we refer the reader to Ref. 47, where the connection between Sobolev spaces and absolute continuity is considered too.

Regarding \(H_s^y\), the defining equation reads
\[
\left( -\frac{\hbar^2}{2m} \partial_x^2 \phi + mgy, \phi \right) = (\phi, H_s^y \phi)
\]
for all \(\phi \in C_0^\infty(\mathbb{R}_+^*)\) and for any \(\psi \in D(H_s^y)\). Equation (22) leads to
\[
\int_0^\infty dy \sqrt{s} \phi(y) \psi(y) = \frac{2m}{\hbar^2} \int_0^\infty dy \sqrt{s} \phi(y) (mgy - H_s^y) \psi(y).
\]
Therefore, the second weak derivative of \(\psi \in D(H_s^y)\) with respect to \(y\) and the action of \(H_s^y\) on \(\psi\) are, respectively, given by
\[
\psi''(y) = \frac{2m}{\hbar^2} (mgy - H_s^y) \psi(y), \quad H_s^y \psi(y) = -\frac{\hbar^2}{2m} \psi''(y) + mgy \psi(y).
\]

Since the gravitational potential energy is continuous on \(\mathbb{R}_+^*\), then it is also locally square integrable, i.e., it is square integrable on every compact subset of \(\mathbb{R}_+^*\). We express this fact by writing \(V(y) \in L^2_{\text{loc}}(\mathbb{R}_+^*)\). Therefore, the product \(V(y) \phi(y)\) is locally square integrable too. Also, the restriction of a square integrable function to a compact subset of \(\mathbb{R}_+^*\) is still square integrable, so \(H_s^y \phi \in L^2(\mathbb{R}_+^*) \subset L^2_{\text{loc}}(\mathbb{R}_+^*)\). Using these facts in Eq. (24), we see that if \(\psi \in D(H_s^y)\), then its second weak derivative \(\psi'' \in L^2_{\text{loc}}(\mathbb{R}_+^*)\). Therefore, the domain of \(H_s^y\) reads
\[
D(H_s^y) = \left\{ \psi \in L^2(\mathbb{R}_+^*); \psi'' \in L^2_{\text{loc}}(\mathbb{R}_+^*), -\frac{\hbar^2}{2m} \psi'' + mgy \psi \in L^2(\mathbb{R}_+^*) \right\}.
\]

from what we see that \(D(H_s^y) = C_0^\infty(\mathbb{R}_+^*) \subset D(H_s^y)\). Consequently, the operator \(H_s^y\) is not self-adjoint.

The results presented above show that neither \(H_s\) nor \(H_s^y\) are self-adjoint. This fact leads us to consider the self-adjoint extensions of \(H\). The intuitive idea behind the theory of self-adjoint extensions is that "the larger the domain of a Hermitian operator, the smaller the domain of its adjoint." In fact, if \(S\) is a Hermitian extension of a Hermitian operator \(T\), then \(D(T) \subset D(S) \subset D(T^*)\). If the action of \(S\) in its domain is formally given by the same action as \(T\), then the task of finding self-adjoint extensions of \(T\) reduces to the one of appropriately choosing the domain of the extension, so that \(D(S)\) turns to be equal to \(D(T^*)\). We notice that, according to the above chain of inclusions, every Hermitian extension of \(H\) will be a restriction of \(H^*\), i.e., the domain of every self-adjoint extension of \(H\) will be obtained from the domain of \(H^*\) itself.

In order to present the theorem that completely classifies the self-adjoint extensions of any Hermitian operator \(T\) acting on a Hilbert space \(\mathcal{H}\), we define the deficiency subspaces of \(T\), denoted by \(\mathcal{K}_+(T)\) and \(\mathcal{K}_-(T)\), as the vector spaces of the square integrable solutions of the equations \(T\psi = i\lambda \psi\) and \(T\psi = -i\lambda \psi\), respectively (the positive real constant \(\lambda\) was introduced only for dimensionality considerations). The deficiency indices are defined by \(n_+ = \dim[\mathcal{K}_+(T)]\) and \(n_- = \dim[\mathcal{K}_-(T)]\). We say that an operator \(T\) is essentially self-adjoint if and only if its closure \(\overline{T}\) is self-adjoint. It turns out that an essentially self-adjoint operator has only one self-adjoint extension, namely, its closure. We also remark that every Hermitian operator is closable. We are now ready to enunciate the von Neumann theorem (for details on the above definitions and results and for the complete version of the von Neumann theorem, see Refs. 40 and 45):
(a) $T$ is essentially self-adjoint if and only if $n_+ = n_- = 0$;
(b) $T$ has self-adjoint extensions if and only if $n_+ = n_- = 0$;
(c) there is a one-to-one correspondence between the unitary maps $U \mathbb{K}_0(T) \to \mathbb{K}_0(T)$ and the self-adjoint extensions of $T$ (which we shall denote by $T_U$);
(d) the domain of $T_U$ is $D(T_U) = \{ \psi + \psi_+ + U \psi_+ \in D(T^*) : \psi \in D(T), \psi_+ \in D(U) \}$.

A few remarks on the consequences of the above theorem are worthy. First of all, we realize that a Hermitian operator may have several self-adjoint extensions, as well as it may have none. In the former case, the analysis of the physical conditions governing the behavior of the system at the boundaries may help one to choose the appropriate self-adjoint extension. We also note that in case $n_+ = n_- = 1$, any self-adjoint extension of $T$ corresponds to a unitary $n \times n$ matrix. That means that the family of self-adjoint extensions is parametrized by $n^2$ real parameters, corresponding to the $n^2$ independent parameters of the Lie group $U(n)$. Finally, we notice that the self-adjoint extensions of a Hermitian operator $T$ are restrictions of $T^*$ to appropriate subsets of $D(T^*)$. Once we know the formal expression of $T^*$, the task of finding the self-adjoint extensions of $T$ reduces to that of finding the domains of self-adjointness of the formal expression of $T^*$.

We now apply the above mathematical results to the task of obtaining the self-adjoint extensions of $H_\Omega$, $H_r$, and $H$. Regarding $H_\xi$, the relevant differential equations, as well as their solutions, are given, respectively, by

$$ - \frac{\hbar^2}{2m} \psi'''_\pm(x) - \frac{mg}{2} i \lambda \psi'_\pm(x) = \pm i \lambda \psi_\pm(x), \quad \psi_\pm(x) = e^{i \omega_\pm x}. $$ (26)

The characteristic equation for each one of the complex constants $\omega_\pm$ is a quadratic equation, which can be readily solved. Hence, all the solutions $\psi_\pm(x)$ are divergent in one or the other of the end points ($\pm \infty$) of the domain of integration. Consequently, all these functions fail to be square integrable on $\mathbb{R}$. Therefore, we have $n_+ = n_- = 0$, which means that $H_\xi$ is essentially self-adjoint. Its closure, $\bar{H}_\xi$, is its unique self-adjoint extension, which implies that (it should be noted that if an operator $T$ and its adjoint $T^*$ are Hermitian operators, then $T^*$ is self-adjoint) $\bar{H}_\xi = H_\xi^*$, so we can write

$$ \bar{H}_\xi = \frac{\hbar^2}{2m} h_\xi^* + \frac{mg \theta}{2} p_\xi, \quad D(\bar{H}_\xi) = D(H_\xi^*), \quad [\text{as given by Eq. (21)}]. $$ (27)

Regarding $H_r$, each one of the equations $H_r \psi = \pm i \lambda \psi$ can be put into the form of an Airy equation in the complex plane, i.e.,

$$ - \frac{\hbar^2}{2m} \psi''_\pm(y) + mg \psi_\pm(y) = \pm i \lambda \psi_\pm(y) \Rightarrow \psi''_\pm(z_\pm) - \left( \frac{2m^2 g}{\hbar^2} \right) z_\pm \psi_\pm(z_\pm) = 0, $$ (28)

where $z_\pm = y \mp i \lambda / (mg)$. The complex Airy equation has two linearly independent complex solutions, the Airy functions, $Ai(z_\pm)$ and $Bi(z_\pm)$, so that the differential equation for $\psi_\pm$ has two linearly independent solutions, $Ai(z_\pm)$ and $Bi(z_\pm)$, and analogously for $\psi_\pm$. In order to investigate the square-integrability of the wave-functions $\psi''_\pm(y) = Ai(z_\pm)$ and $\psi''_\pm(y) = Bi(z_\pm)$ on $\mathbb{R}_+$, we look at their behavior near the boundaries of the domain of integration. As the potential is well behaved at the origin, we only have to care about the asymptotic behavior of $Ai(z_\pm)$ and $Bi(z_\pm)$ for $|z_\pm| \to \infty$ [indeed, for $y = \text{Re}(z_\pm) \to \infty$]. Since $\text{Re}(z_\pm) > 0$ and $\text{Im}(z_\pm)$ is a nonzero complex constant, the complex variables $z_\pm$ belong to the sector of the complex plane defined by $|\text{Arg}(z_\pm)| \leq \pi - \delta$ for some $\delta > 0$ (the variables $z_\pm$ never reach the negative $Ox$-axis). In this case, the asymptotic expressions read
\[ Ai(z_\pm) \sim \frac{1}{2\sqrt{\pi}} z_\pm^{-1/4} e^{-(2/3)z_\pm^{3/2}}, \quad Bi(z_\pm) \sim \frac{1}{\sqrt{\pi}} z_\pm^{-1/4} e^{(2/3)z_\pm^{3/2}}. \]  

One can see that, in both cases \((\pm i\lambda)\), only the solutions \(\psi_\pm(y) = Ai(z_\pm)\) are square integrable in the domain of integration \(\mathbb{R}_+ = (0, \infty)\), which means that the deficiency indices of the operator \(H_y\) are both equal to 1. Thus, since \(n_+ = n_- = 1\), it follows that the family of self-adjoint extensions of \(H_y\) is parametrized by just one real number.

Let \(S\) be a self-adjoint extension of \(H_y\) with domain \(D(S) \subset D(H_y^0)\). We know from Theorem 1 (see also the remarks that followed the theorem) that any self-adjoint extension of \(H_y\) is a restriction of \(H_y^0\), i.e., \(S\psi = H_y^0\psi, \quad \forall \psi \in D(S)\). The domain \(D(S)\) is characterized as a subset of \(D(H_y^0)\) whose elements fulfill a particular set of boundary conditions. In order to find these boundary conditions, we make use of the Hermiticity of \(S\) and the fact that this operator is a restriction of \(H_y^0\). As a result, we have

\[ (H_y^0 \phi, \psi) = (\phi, H_y^0 \psi) \]  

for all \(\phi, \psi \in D(S)\), what leads to [see Eq. (24)]

\[ \int_0^{\infty} dy \overline{\psi(y)} \phi(y) = \int_0^{\infty} dy \overline{\psi(y)} \phi(y), \quad \forall \phi, \psi \in D(S). \]  

Notice that the potential terms were readily cancelled out. Then, performing integration by parts on both sides of Eq. (31), we find

\[ \overline{\psi}(\infty) \phi(\infty) - \overline{\psi}(0) \phi'(0) = \overline{\psi}(0) \phi(0) - \overline{\psi}'(0) \phi(0). \]  

We now make use of the fact that the functions belonging to \(D(H_y^0)\), as defined in Eq. (25), vanish at infinity (see Ref. 51 for the demonstration of this result). It results that

\[ \frac{\overline{\psi}'(0)}{\overline{\psi}(0)} = \frac{\phi'(0)}{\phi(0)}. \]  

Equation (33) is fulfilled if we impose the boundary condition \(\psi(0) = \alpha \psi'(0), \quad \alpha \in \mathbb{R}\), for all functions belonging to \(D(S)\). This fact leads us to modify the notation used in the above considerations and write \(H_{y,\alpha}\) in place of \(S\). Therefore,

\[ D(H_{y,\alpha}) = \{ \psi \in D(H_y^0); \psi(0) = \alpha \psi'(0) \}. \]  

The range of \(\alpha\) can be extended so as to comprehend the case \(\alpha = \infty\), which corresponds to \(\psi'(0) = 0\).

The self-adjoint boundary conditions in Eq. (34) can be recognized as the canonical self-adjoint boundary conditions of the Sturm–Liouville problem for the operator \(H_y\) (see, e.g., Ref. 52).

Let us finally calculate the energies of the particle in the gravitational quantum well. As usual, the energies are given by the eigenvalues of the Hamiltonian operator of the system. But according to Eq. (34), there exists a whole family of mathematically allowed self-adjoint Hamiltonians which could be associated with a quantum particle moving under the action of a uniform gravitational potential.

The infinite forces experienced by the particle at the perfectly reflecting mirror, situated at the bottom of the well, lead us to impose the condition \(\psi(0) = 0\). This boundary condition corresponds to the choice \(\alpha = 0\) in Eq. (34). Therefore, the natural boundary condition suggested by the analysis of the physics of the problem has selected one of the possible self-adjoint extensions of the Hamiltonian \(H_y\) and, therefore, of the full Hamiltonian \(H\). The other self-adjoint extensions obtained in Sec. III B correspond to situations in which partial reflection occurs at the mirror. The boundary conditions corresponding to those extensions may be relevant to further investigations on the phenomenology of the gravitational quantum well.
We can now proceed to solve the energy eigenvalue problem. Since the motion is free along $Ox$, we use the ansatz $\phi(x,y) = e^{ikx}\phi(y)$ into (13) to get

$$\frac{d^2 \phi}{dy^2} + \frac{2m^2 g}{\hbar^2} \left( \frac{\mathcal{E}_\theta}{mg} \right) y \phi(y) = 0,$$

where

$$\mathcal{E}_\theta = E - \frac{\hbar^2 k^2}{2m} - \frac{mgk\theta}{2}.$$  \hfill (36)

Now, by setting

$$\xi = \frac{y - b_\theta}{a}, \quad a = \left( \frac{\hbar^2}{2m^2 g} \right)^{1/3}, \quad b_\theta = \frac{\mathcal{E}_\theta}{mg},$$

we can put (35) in the form of an Airy equation,

$$\frac{d^2 \phi(\xi)}{d\xi^2} - \xi \phi(\xi) = 0,$$

whose general solution is

$$\phi(y) = \text{Ai}\left(\frac{y - b_\theta}{a}\right) + B\text{Bi}\left(\frac{y - b_\theta}{a}\right).$$ \hfill (39)

We now simply impose the adequate boundary conditions for $\phi(y)$,

$$\phi(0) = 0, \quad \lim_{y \to \infty} \phi(y) = 0,$$

that leads to

$$\phi(y) = \text{Ai}\left(\frac{y - b_\theta}{a}\right).$$ \hfill (41)

The application of the boundary condition of vanishing wave-function at the mirror leads to $\text{Ai}\left(\frac{b_\theta}{a}\right) = 0$, so that $-b_\theta/a$ are the roots of the Airy function $\text{Ai}$, i.e.,

$$b_{\theta,n} = -a\alpha_n,$$ \hfill (42)

where $\alpha_n$ denotes the $n$th zero of $\text{Ai}$. Result (42) combined with the definition of $b_\theta$ [see Eqs. (36) and (37)] gives the spectrum of the Hamiltonian,

$$E_{k,n,\theta} = \frac{\hbar^2 k^2}{2m} + \left( \frac{mg^2 \hbar^2}{2} \right)^{1/3} \left( -\alpha_n \right) + \frac{mgk\theta}{2}.$$ \hfill (43)

For $\theta = 0$, the result found in ordinary quantum mechanics is recovered. We note that the effect of noncommutativity also disappears for $k=0$, what simply corresponds to a one-dimensional movement, along the direction of the gravitational field.

We remark that the energy spectrum of the noncommutative gravitational quantum well is invariant under parity. Indeed, recall that in two dimensions, the matrix element of the parity operator is given by $P_{ij} = -\epsilon_{ij}$, where $\epsilon_{ij}$ is the Levi–Civita symbol. Thus, the noncommutativity matrix transforms as $\theta'_{ij} = -\epsilon_{ij}$, that is, $\theta' = \theta_{12} = -\theta$. On the other hand, since the momentum operator $\hat{p}_x$ anticommutes with $\hat{P}$, it follows that the parity transformation implies that $k$ goes into $-k$, thus showing that the spectrum in Eq. (43) is invariant under $\hat{P}$. It is important to notice that if we had adopted the three-dimensional scenario for the problem of the quantum well, we would...
have found the same result, since the parity transformation in three dimensions changes the sign of \( g \), while keeping \( \theta_{ij} \) unchanged. In any case the expression of the energy is unaffected by \( P \), a result which obviously follows from the fact that \( H \) commutes with \( P \).

We finish this section by analyzing what is, now in the noncommutative case, the analog of the classical turning point. Just like in the ordinary case, it corresponds to the vertical position from where the wave-functions start to fall off exponentially. From Eq. (37) we see that \( \xi=0 \) implies

\[
y = b_\theta = \frac{E - \frac{\hbar^2 k^2}{2m} - \frac{mg \theta}{2}}{mg} = b - \frac{k \theta}{2},
\]

where \( b \) denotes the classical turning point. Thus, we see that for \( \theta \neq 0 \) the turning point position acquires a \( \theta \)-dependent correction, a result that is compatible with the corresponding noncommutative deformation of the classical equations of motion.

IV. THE GRANIT EXPERIMENT: AN UPPER-BOUND ON \( \theta \)

Having found the bound states of a particle in a noncommutative gravitational quantum well, we can now constrain the value of the noncommutative parameter, by resorting to data of the experiment performed by Nesvizhevsky et al.—the GRANIT experiment.\(^{18,19}\) In this experiment the lowest quantum bound states of neutrons on free-fall in the Earth’s gravitational field were observed and their energy determined. Due to the weaker strength of gravity, as compared to nuclear and electromagnetic interactions, it is very difficult to perform experiments in which gravity plays a role in the manifestation of the quantum nature of matter. Thus, the GRANIT experiment, performed under challenging, extremely sensitive experimental designed conditions, is another striking observation of the wave behavior of matter in a gravitational field. (An earlier experiment in which the quantum, wave nature of matter was manifested, due to its interaction with the Earth’s gravitational field, is that of the observation of gravitationally induced quantum interference of neutrons.\(^{33}\) The data of this experiment could also be used to constrain the value of \( \theta \).) This was achieved by means of a gravitational quantum well, formed by the Earth’s gravitational field, and a horizontal reflecting mirror (considered as perfectly reflecting). Due to their charge neutrality, long lifetime, and low mass, neutrons are suitable to perform this kind of experiment, in which the effect of the interactions other than gravity must be negligible. A horizontal beam of ultracold neutrons was thus allowed to fall freely, flying above the reflecting mirror at the bottom. As no forces act on neutrons horizontally and just gravity acts vertically, we have a gravitational potential well along the latter direction. For details on the experimental setup, see Refs.\(^{18} \) and \(^{19} \).

An upper-bound on \( \theta \) can be established by imposing that the \( \theta \)-dependent corrections to the energy implied by Eq. (43) be smaller or of the order of the maximum differences of the energy levels provided by the GRANIT experiment, i.e., according to its error bars. Hence we require that

\[
\frac{mg \theta}{2} \leq \Delta E_{\text{exp}},
\]

so that

\[
\theta \leq \frac{2 \Delta E_{\text{exp}}}{mgk}.
\]

For the first two energy levels, the experiment gives \( \Delta E_{1\text{exp}}=6.55 \times 10^{-32} \) J (\( n=1 \)) and \( \Delta E_{2\text{exp}}=8.68 \times 10^{-32} \) J (\( n=2 \)). For neutrons we have \( m \approx 1.675 \times 10^{-27} \) kg, and in the experiment,
the neutrons had a mean horizontal speed \( \langle v_x \rangle = 6.5 \text{ m/s} \) so that \( k = \langle p_x \rangle / \hbar = m \langle v_x \rangle / \hbar = 1.03 \times 10^8 \text{ m}^{-1} \). Then, by considering \( g = 9.81 \text{ m/s}^2 \) we arrive at the following upper-bounds for \( \theta \):

\[
\theta \leq 0.771 \times 10^{-13} \text{ m}^2 \quad (n = 1),
\]

\[
\theta \leq 1.021 \times 10^{-13} \text{ m}^2 \quad (n = 2).
\]

We point out that the literature on the noncommutative gravitational well that resorts to data of the GRANIT experiment does not arrive at an upper-bound on \( \theta \). As they considered noncommutativity in both configuration and momentum spaces, they have found an upper-bound on the parameter of momentum noncommutativity instead.\(^{20-23}\) Since we consider spatial noncommutativity only, we can establish an upper-bound on the parameter of spatial noncommutativity. We remark that other upper-bounds on \( \theta \) were established by experimental data related to the Lorentz invariance\(^{54}\) and to the Lamb shift,\(^{55}\) for instance.

One of the most important prospects about the GRANIT experiment is the improvement of the energy resolution of the neutrons energy levels. In principle, with the energy resolution \( \Delta E \) limited only by the uncertainty principle \( (\Delta E \Delta \tau \sim \hbar) \), one could achieve a value as low as \( 10^{-18} \text{ eV} \), if \( \Delta \tau \) approaches the neutron lifetime. Improvement in the energy resolution could help in testing the proportionality between inertial and gravitational masses for neutrons.\(^{19}\) The equivalence principle in the context of the gravitational quantum well was studied in Ref.\(^{56}\) but in a manner completely different from that we consider in Sec. V. In fact, in the following, we do not need even to consider that a well is set up, the equivalence principle being studied by means of the time of flight of a quantum particle.

V. QUANTUM CLOCK AND TIME OF FLIGHT MEASUREMENT

By resorting to the basic notion that “a clock is a dynamical system which passes through a succession of states at constant time intervals,”\(^{32}\) that is, that “… the measurement of time actually is the observation of some dynamical variable, the law of motion of which is known…,”\(^{32}\) Peres has modeled a simple quantum clock by means of a quantum rotor (also known as the Larmor clock). When coupled to another system, this can measure the duration of physical processes as well as keep a permanent record of it, such as the time of flight of a particle, or even to control the duration of a physical process\(^{32}\) (see also Ref.\(^{57}\)).

In order to address the essential features involved in modeling a quantum clock, and mainly to prepare the discussion in Sec. VI, in the following we briefly review Peres construction of a quantum clock and its particular use in a time of flight experiment involving the motion of a free quantum particle, as considered in Ref.\(^{32}\).

Peres basic idea in modeling a quantum clock is to consider a quantum rotor, since this can be regarded as describing the motion of a pointer on a clock dial. Thus, he considered a clock with an odd number of “pointer states,” \( N = 2j + 1 \), represented by

\[
u_n(q) = \frac{1}{\sqrt{2\pi}} e^{inq}, \quad n = -j, \ldots, j,
\]

where \( q \) is the clock’s degree of freedom and \( 0 \leq q < 2\pi \).

Another suitable orthogonal basis of states for the clock is
\[ u_\kappa = \frac{1}{\sqrt{N}} \sum_{n=-j}^{j} e^{-i2\pi n/N} u_n(q), \]

\[ \sigma = \frac{1}{\sqrt{2\pi N}} \frac{\sin N/2}{\sin q/2} \left( \frac{2\pi \kappa}{N} \right) \]

where \( \kappa = 0, \ldots, N-1 \). It follows that for large values of \( N \) the state \( u_\kappa(q) \) is sharply picked at \( 2\pi \kappa/N \). We can refer to the states \( u_\kappa \) as the clock states. By defining the projection operators \( P_\kappa v_n = \delta_{n\kappa} v_n \), a “clock time” operator can be defined as

\[ T_c = \tau \sum_\kappa \kappa P_\kappa, \]

where \( \tau \) is the time resolution of the clock. Its eigenstates are \( u_{\kappa\tau} \) with \( t_\kappa = \kappa \tau \) as the corresponding eigenvalues.

The clock’s Hamiltonian is

\[ H_c = \omega J, \quad \omega = 2\pi/(N\tau), \quad J = -ih \frac{\partial}{\partial q}. \]

The eigenstates of \( H_c \) are the vectors \( u_n(q) \), defined in Eq. (49). The eigenvalue corresponding to \( u_n(q) \) is \( nh\omega \). From (49) and (52), it follows that

\[ e^{-iHc\tau} u_n(q) = u_{n+1(\text{mod} N)}(q). \]

Assuming that \( u_0 \) is the initial state of the clock, the above result implies that the clock will pass successively through the states \( u_0, u_1, u_2, \ldots \), at time intervals given by \( \tau \).

Before we apply the Peres quantum clock model to our case of interest, first it is very instructive to recall its application in determining the time of flight of a free particle traveling between two assigned points. One has to demand that the clock be activated when the particle pass by the first point and then be stopped when it passes by the second one. Let us consider that the particle moves along the direction \( Ox \), so that \( x_1 \) and \( x_2 \) correspond to the two positions of interest. In the Schrödinger representation, the Hamiltonian for the composed, particle-clock system reads

\[ H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - i\hbar \omega \Theta(x-x_1)\Theta(x_2-x) \frac{\partial}{\partial q}, \]

so that the clock runs only if the particle lies within the interval \([x_1, x_2]\).

Following Peres, we set the initial state of the clock as \( u_0 = \sum_n u_n/\sqrt{N} \). As \( H \) and \( H_c = \omega J \) commute, it is simpler to solve the equation of motion for the clock in an eigenstate of \( J \), i.e., \( u_n \), and then sum over these partial solutions to get the evolved wave-function corresponding to the initial state \( u_0 \). Therefore, the initial state of the (still not coupled) particle-clock system can be written as \( \Psi_0(x,q)e^{-iE\tau/\hbar} \), with \( \Psi_0(x,q) \) given in the factorized form (see Refs. 32 and 58),

\[ \Psi_0(x,q) = A_k e^{ikx} v_0(q) = A_k e^{ikx} \frac{1}{\sqrt{N}} \sum_{n=-j}^{j} u_n(q), \]

where \( E \) is the energy of the particle, \( k = \sqrt{2mE}/\hbar \), and \( A_k \) is a normalization constant.

On the other hand, the final state of the system cannot be factorized as \( \Psi_0(x,q) \), since the motion of the particle in the region \( x_1 < x < x_2 \) will activate the clock, making the particle and clock coordinates to mix. Thus, we write
\[ \Psi(x,q) = \frac{1}{\sqrt{N}} \sum_{n=-j}^{j} \psi_n^k(x) u_n(q). \] (56)

The substitution of (56) into the Schrödinger equation leads to

\[-\frac{\hbar^2}{2m} \frac{d^2\psi_n}{dx^2} + [\hbar \omega \Theta(x-x_1) \Theta(x_2-x) - E] \psi_n^k(x) = 0. \] (57)

Thus, we see that \( \psi_n^k \) satisfies the Schrödinger equation for a particle under the action of a rectangular barrier of height \( V_n = \hbar \omega \) and length \( L = x_2 - x_1 \). Outside the barrier we have (we are now basically following Leavens presentation of the original Peres quantum clock model\(^58\))

\[ \psi_n^k(x) = \begin{cases} e^{ikx} + R_n(k)e^{-ikx}, & x < x_1 \\ T_n(k)e^{ikx}, & x > x_2, \end{cases} \] (58)

where \( T_n(k) \) and \( R_n(k) \) are the transmission and reflection amplitudes, respectively, given by

\[ T_n(k) = \frac{e^{-ikL}}{\cos(k_nL) - \frac{i}{2} \left( \frac{k}{k_n} + \frac{k_n}{k} \right) \sin(k_nL)} \] (59)

and

\[ R_n(k) = \frac{\frac{i}{2} \left( \frac{k_n}{k} - \frac{k}{k_n} \right) e^{2ikx_1} \sin(k_nL)}{\cos(k_nL) - \frac{i}{2} \left( \frac{k_n}{k} + \frac{k}{k_n} \right) \sin(k_nL)}, \] (60)

and where \( k_n = \sqrt{2m(E-n\hbar \omega)/\hbar} \).

Of course the particle will be perturbed when coupled to the physical clock. In order that one has a small disturbance on the particle evolution, one must consider that such coupling is sufficiently small. We therefore set \( V_n \ll E \), from which we have

\[ k_n = k - n\omega(2E/m)^{-1/2} = k - n\omega(\hbar k/m). \] (61)

Furthermore, in the limit of small coupling, we also have \( T_n(k) \approx \exp(i(k_n-k)L) \), so that \( |T_n(k)| \approx 1 \). Under the above conditions we can write the phase shift due to the clock as

\[ (k_n-k)L = -n\omega L(2E/m)^{-1/2} = \frac{n\omega L}{\hbar k/m}. \] (62)

Let us call \( T = L/((\hbar k/m)) \). Now we can express the final wave-function for the particle-clock system as

\[ \Psi(x,q) \approx A_k e^{ikx} \frac{1}{\sqrt{N}} \sum_{n=-j}^{j} e^{-in\omega T(\hbar k/m)} u_n(q), \] (63)

that is,

\[ \Psi(x,q) \approx A_k e^{ikx} \frac{1}{\sqrt{N}} \sum_{n=-j}^{j} e^{-in\omega T} u_n(q) = A_k e^{ikx} v_0(q - \omega T). \] (64)

Since \( e^{-i\hat{H}_T\tau} u_n(q) = u_n(q - \omega T) \), it follows that the pointer, initially directed to \( q = 0 \), will be found directed to \( q = \omega T \) after the particle leaves the region where the clock runs. By noting that \( v = \sqrt{2E/m} = \hbar k/m \) is the velocity of a free quantum particle with energy \( E \), we see that the Peres
clock records the time of flight of a particle which travels between two assigned points. The key fact is that the time of flight is encoded in the phase shift due to the clock barrier.

We remark that the quantum clock does not measure the absolute instants of time in which the particle passes through the positions \( x_1 \) and \( x_2 \), but only the time difference to travel between them. This fact avoids the collapse of the particle wave-function. This is the advantage of using a quantum clock to study delocalized states, as energy eigenstates (see Sec. VI).

**VI. TIME OF FLIGHT IN A UNIFORM GRAVITATIONAL FIELD IN NCQM**

In Ref. 31, Davies applied the Peres model of a quantum clock to determine the time of flight of a quantum particle in its round trip in a uniform gravitational field, when it is vertically projected up. Davies' interest in studying this sort of “quantum Galileo experiment” was motivated by asking if there would be a violation of the (weak) equivalence principle when one considers quantum particles, as they may tunnel into the classically forbidden region of the gravitational potential. Therefore, a mass-dependent delay in the time of flight might result, as compared to the classical case. As Davies remarks, the answer to this question is not a priori obvious in the case one is handling with energy eigenstates, which are delocalized states, with no corresponding classical counterparts on localized bodies (for a study of the motion of wave packets and the “quantum Galileo experiment,” see Ref. 59). Davies found, by assuming the equality of the inertial \( (m_i) \) and gravitational \( (m_g) \) masses, that when the measurement is made far from the classical turning point, the time of flight is equal to the classical result. Therefore, in this sense, the (weak) equivalence principle holds in quantum mechanics, even when one considers highly nonclassical states.

Inspired by Davies, we next apply the quantum clock model of Peres to determine the time of flight of a particle in a uniform gravitational field in the context of NCQM, in order to address the “quantum Galileo experiment” and the question of the validity of the equivalence principle in NCQM. We will assume that \( m_i = m_g = m \).

We now make use of the simple clock model of Sec. V to measure the interval of time during which the particle lies within the semi-infinite region of space defined by \( y_0 \leq y < \infty \), where \( y_0 \) is some fixed height, at where we can consider that the clock is set up. We consider that the particle is obliquely projected up from some height below \( y_0 \).

Let \( P_{y_0} \) be the projection operator defined by

\[
P_{y_0} \psi(x,y) = \Theta(y-y_0) \psi(x,y).
\]

The vertical particle degree of freedom couples to that of the quantum clock through the interaction Hamiltonian,

\[
H_I = P_{y_0} \otimes H_c = \Theta(y-y_0) \omega J.
\]

The action of the total Hamiltonian on wave-functions reads

\[
\hat{H}\Psi = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} - \frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial y^2} + mg\Psi + \Theta(y-y_0)(\omega J \Psi).
\]

We now consider the noncommutative analog of the above problem. We know that the action of the potential energy on noncommutative wave-functions can be written in terms of the Moyal product, \( V \star \psi \). In order to calculate this function, we note that there are two different contributions to the total interaction. For the gravitational potential, we can directly apply Eq. (5), thus obtaining the following exact result:

\[
mgy \star \Psi(x,y) = mg \left( y\Psi(x,y) - \frac{i\theta}{2} \frac{\partial \Psi}{\partial x} \right).
\]

Denoting the energy of the particle-clock system by \( E \), its eigenstate is given by \( \Psi(x,y,q)e^{-iEt/\hbar} \). From Eqs. (7) and (68), we thus have

\[
\hat{H}\Psi = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} - \frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial y^2} + mg\Psi + \Theta(y-y_0)(\omega J \Psi).
\]
where we have used the fact that the operator $H = \omega J$ commutes with the $\ast$-product. Further, by making use of the **ansatz** $\Psi(x,y,q)e^{ikx}\Phi(y,q)$ into (69), we can write

$$\frac{-\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial y^2} + mg\Psi + \omega J\Theta(y-y_0) \ast \Psi = E\Psi,$$

(70)

It remains to calculate the product $\Theta(y-y_0) \ast \Psi$. Since $\Theta(y-y_0)$ is not a differentiable function, we cannot directly apply the definition of the Moyal product to this case. In order to overcome this problem, we have to interpret the term $\Theta(y-y_0) \ast \Psi$ in the sense of generalized functions (distributions). We start by recalling that the Heaviside function $\Theta(y-y_0)$ can be used to define the Heaviside distribution, $\Theta: C^\infty_0(\mathbb{R}) \rightarrow C$, given by

$$\Theta(\phi) = \int_{\mathbb{R}} d\phi \Theta(y-y_0) \phi(y).$$

(71)

We say that the function $\Theta(y-y_0)$ represents the regular distribution $\Theta$. Even though the Heaviside function is not differentiable, the corresponding distribution $\Theta$ is infinitely differentiable. Indeed, recall that given a distribution $T$, its $n$th derivative (in the sense of distributions) is defined by

$$(D^{(n)}T)(\phi) = (-1)^n T \left( \frac{d^n \phi}{dy^n} \right)$$

(72)

for all $\phi \in C^\infty_0(\mathbb{R})$.

Our aim is to construct the $\ast$-product between a distribution and an ordinary function. In order to do this we will use Eq. (5) as a guide and apply the definition given by Eq. (72), so as to make sense of $\partial_x \Theta$ and $\partial_y \Theta$. Since the Heaviside distribution in Eq. (71) acts on functions of the variable $y$ only (the test functions $\phi(y)$), and bearing in mind Eq. (72), we consider the following prescription:

$$\frac{\partial}{\partial x} \Theta \rightarrow 0, \quad \frac{\partial^n}{\partial y^n} \Theta \rightarrow D^{(n)} \Theta.$$  

(73)

Thus, in analogy with Eq. (5) we can write

$$\Theta \ast \Psi = \Psi(x,y,q) \sum_{n=0}^\infty \frac{1}{n!} \left( \frac{k\theta}{2} \right)^n D^{(n)} \Theta,$$

(74)

Equation (74) shows that the Moyal product $\Theta \ast \Psi$ is equivalent to the product of an ordinary function ($\Psi$) by another distribution.

In order to achieve a suitable representation for $\Theta \ast \Psi$, we apply Eq. (74) on a generic test function $\phi \in C^\infty_0(\mathbb{R})$. Recalling that the product $uT$ between an ordinary function $u$ and a distribution $T$ is the distribution defined by $(uT)(\phi) = T(u\phi)$, it results that

$$\left( (\Theta \ast \Psi) \phi \right) = \sum_{n=0}^\infty \frac{1}{n!} \left( -\frac{k\theta}{2} \right)^n \int_{\mathbb{R}} d\phi \Theta(y-y_0) \frac{d^n}{dy^n} \left( \Psi(x,y,q) \phi(y) \right),$$

(75)

what, after the change of variable $\xi = y - k\theta/2$, leads to
distribution associated with a possible time-reversal symmetry breaking in our noncommutative model.

Equation (76) permits us to identify the function \((\Theta \star \Psi)(x,y,q)\) which represents the regular distribution \(\Theta \star \Psi\) [in the same sense as the function \(\Theta(y-y_0)\) represents the regular distribution \(\Theta\)].

\[
(\Theta \star \Psi)(x,y,q) = \Theta(y-y_0)\Psi(x,y,q),
\]

where \(y_0 = y_0 - k\theta/2\).

Now, replacing the term \(\Theta(y-y_0)\star\Psi(x,y,q)\) in Eq. (70) by Eq. (77), we obtain

\[
-\frac{\hbar^2}{2m} \frac{\partial^2 \Phi}{\partial y^2} + mg\Psi + \Theta(y-y_0)\omega \Phi = \left( E - \frac{\hbar^2 k^2}{2m} - \frac{mgk\theta}{2} \right) \Phi,
\]

where we have made use of the ansatz \(\Psi(x,y,q) = e^{ikx}\Phi(y,q)\).

We remark that the above construction is well-defined only if \(k\theta > 0\). Indeed, Eq. (75) implies that \(\psi\) must be infinitely differentiable in \((y_0, \infty)\). But according to Eq. (78), the function \(\frac{\partial \Phi}{\partial y}\) is discontinuous at \(y = y_0 - \frac{k\theta}{2}\), so that if \(k\theta < 0\), then \(\psi\) is not smooth in \((y_0, \infty)\). This result could be associated with a possible time-reversal symmetry breaking in our noncommutative model (the effect of a time reversal can be obtained by the replacement of \(k\) by \(-k\) in the equations of the model, the other parameters kept unchanged). We note that at the level of NCQFTs, the CPT symmetry is still preserved, but with individual violations of the C, P, and T symmetries. In any case, further studies are necessary in order to properly generalize the definition given in Eq. (74).

It is now useful to write the wave-function \(\Phi(y,q)\) just like in (56), i.e.,

\[
\Phi(y,q) = \frac{1}{\sqrt{N_{n=q}}} \sum_j \psi_j^q(y) u_n(q),
\]

where \(u_n(q)\) are the basis vectors defined in (49). Substituting (79) into (78) we find

\[
\frac{1}{\sqrt{N_{n=q}}} \sum_j \left( -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_j^q}{\partial y^2} + \left[ mg + n\hbar \omega \Theta(y-y_0) - \mathcal{E}_n \right] \psi_j^q(y) \right) u_n(q) = 0,
\]

where

\[
\mathcal{E}_n = E - \frac{\hbar^2 k^2}{2m} - \frac{mgk\theta}{2}.
\]

But since the vectors \(u_n\) are linearly independent, we can write a differential equation for each \(\psi_j^q(y)\),

\[
-\frac{\hbar^2}{2m} \frac{\partial^2 \psi_j^q}{\partial y^2} + \left[ mg + n\hbar \omega \Theta(y-y_0) - \mathcal{E}_n \right] \psi_j^q(y) = 0.
\]

There are two different regions to consider, according to the value assumed by the Heaviside function. For \(y < y_0\) we have \(\Theta(y-y_0) = 0\), so that the Schrödinger equation reduces to

\[
\frac{d^2 \psi_j^q(y)}{dy^2} + \frac{2m^2 g}{\hbar^2} \left( \frac{\mathcal{E}_n}{mg} - y \right) \psi_j^q(y) = 0,
\]

whose general solution [see Eqs. (35)–(39)] is
where \( a \) and \( b_\theta \) were defined by Eqs. (37) and (44), respectively.

For \( y \geq y_\theta \) we have \( \Theta(y-y_\theta) = 1 \), so that the Schrödinger equation reads

\[
\frac{d^2 \psi_n^\theta(y)}{dy^2} + \frac{2m^2g}{\hbar^2} \left( \frac{\mathcal{E}_n}{mg} - y \right) \psi_n^\theta(y) = 0,
\]

where

\[
\mathcal{E}_n^\theta = \mathcal{E}_\theta - n\hbar\omega
\]

and \( \mathcal{E}_\theta \) was defined by Eq. (36).

Equation (85) is analogous to Eq. (83), the only difference being in the value of the energy. It follows that its general solution is a linear combination of Airy functions too. Taking this into account, we write the general solution of the Schrödinger equation for \( y \geq y_\theta \) as

\[
\psi_n^\theta(y) = CA_i \left( \frac{y-b_\theta}{a} \right) + BB_i \left( \frac{y-b_\theta}{a} \right),
\]

where

\[
b_\theta = \frac{\mathcal{E}_\theta}{mg} = b_\theta - \frac{n\hbar \omega}{mg}.
\]

The complex constants in (84) and (87) are determined by the matching conditions at \( y = y_\theta \).

By setting \( \xi = (y-b_\theta)/a \), we see that

\[
y = y_\theta \Rightarrow \xi = \frac{y_\theta - b_\theta}{a} = \frac{y_0 - b}{a}.
\]

In what follows we denote this particular value of \( \xi \) by the symbol \( \xi_0 \). Thus,

\[
AA_i(\xi_0) + BB_i(\xi_0) = CA_i \left( \xi_0 + \frac{n\hbar \omega}{mg} \right),
\]

\[
A \frac{dA_i}{d\xi}(\xi_0) + B \frac{dB_i}{d\xi}(\xi_0) = C \frac{dA_i}{d\xi} \left( \xi_0 + \frac{n\hbar \omega}{mg} \right). \tag{90}
\]

By solving the above system we find the values of \( \frac{B}{A} \) and \( \frac{C}{A} \),

\[
\frac{B}{A} = \frac{C}{A} = \frac{1}{\pi} \frac{1}{A_i \left( \xi_0 + \frac{n\hbar \omega}{mg} \right)} \left( \frac{dA_i}{d\xi}(\xi_0) - \frac{dB_i}{d\xi}(\xi_0) \right). \tag{92}
\]

The constant \( A \) remains undetermined. We will choose its value later.
At this point it is useful to put together the results just obtained and write the complete solution of (69). At first we recall that \( \Psi(x,y,q) = e^{ikx} \Phi(y,q) \). Besides that, \( \Phi \) was decomposed according to (79). We thus write

\[
\Psi_{E,k}(x,y,q) = C_{E,k} \sum_{\mathcal{V}} \psi^E_n(y) \psi^q_n(q),
\]

where

\[
\psi^E_n(y) = \begin{cases} 
AAi \left( \frac{y-b_\theta}{a} \right) + BBi \left( \frac{y-b_\theta}{a} \right), & \text{if } y \leq y_\theta \\
CAi \left( \frac{y-b_\theta}{a} \right), & \text{if } y \geq y_\theta,
\end{cases}
\]

with \( B/A \) and \( C/A \) given by (91) and (92). \( C_{E,k} \) is a normalization constant.

We interpret the eigenstate \( \Psi \) as the state of a particle with definite energy, traveling within a region of uniform gravitational field and interacting with a quantum clock. In order to determine the time of flight, we follow the ideas presented in Sec. V and suppose that the interaction between the particle and the clock barrier can be treated as a small perturbation. At this point it is interesting to compare the present situation with that of Sec. V. There, the scattered (transmitted) wave-function could be written as a product of a free particle wave-function by a clock state carrying the information about the time of flight. In the gravitational case the situation is analogous, since that in the limit of small perturbations the only effect of the interaction particle-clock is to make the clock run and record the time of flight. In the following, we will show that in the "far future" the scattered (reflected) wave-function can be written as a product of a wave-function of a particle under the action of a purely gravitational potential (without the clock) by a clock state showing the time of flight.

We now turn our attention to the reflected wave-function in the "far future". In the stationary framework, that means that we must choose \( \xi < 0 \) and \( |\xi| \) large. This fact allows us to make use of the asymptotic form of the Airy functions in (94). If \( \xi < 0 \) and \( |\xi| \) is sufficiently large, we have (see, e.g., Ref. 49)

\[
Ai(\xi) \sim \frac{1}{\sqrt{\pi}} |\xi|^{-1/4} \cos \left( \frac{2}{3} |\xi|^{3/2} - \frac{\pi}{4} \right),
\]

\[
Bi(\xi) \sim \frac{1}{\sqrt{\pi}} |\xi|^{-1/4} \sin \left( \frac{2}{3} |\xi|^{3/2} - \frac{\pi}{4} \right).
\]

The trigonometric functions can be written in terms of complex exponentials, and the reflected wave-function comes from the exponentials with negative exponents. Using (95) and (96) into (84), we obtain the following expression for the reflected wave-function:

\[
\psi^r_{k,ref}(y) \sim A \frac{1}{2\sqrt{\pi}} |\xi|^{-1/4} \left( 1 + i \frac{B}{A} \right) e^{-i\left(2/3\xi^{3/2} - \pi/4\right)}.
\]

Analogously, the incident wave-function is given by

\[
\psi^i_{k,inc}(y) \sim A \frac{1}{2\sqrt{\pi}} |\xi|^{-1/4} \left( 1 - i \frac{B}{A} \right) e^{i\left(2/3\xi^{3/2} - \pi/4\right)}.
\]

We now choose \( A \) in such a way that \( A = 1 + iB \). With this choice the function \( \psi^r_k(y) \) reduces to \( A_i \left( \frac{y-y_\theta}{a} \right) \) in the limit of a negligible clock barrier. As a consequence, the reflected and the incident waves read...
The absence of the barrier, we have calculate the first order correction, denoted by stationary state is the Airy function where the phase shift

\[
\phi_R = \arctan \left( \frac{2B}{A} \frac{1 - \frac{B^2}{A^2}}{1 - \frac{B^2}{A^2}} \right) .
\]  

(102)

Writing the complex constant \( R = (1 + iB/A)/(1 - iB/A) \) in the exponential form, i.e., \( R = |R|e^{i\phi_R} = e^{i\phi_R} \), we get

\[
\psi_{k,\text{ref}}(y) \sim \frac{1}{2\sqrt{\pi}} |\xi|^{-1/4} e^{-i((2/3)|\xi|^3/2 - \pi/4)} ,
\]

\[
(99)
\]

\[
\psi_{\text{inc}}(y) \sim \frac{1}{2\sqrt{\pi}} |\xi|^{-1/4} e^{i((2/3)|\xi|^3/2 - \pi/4)} .
\]

(100)

where the phase shift \( \phi_R \) can be expressed as

\[
\phi_R = \arctan \left( \frac{2B}{A} \frac{1 - \frac{B^2}{A^2}}{1 - \frac{B^2}{A^2}} \right) .
\]

We already know the zeroth order term in the power series expansion \( (\phi_R(0) = 0) \). Let us now calculate the first order correction, denoted by \( \phi_R^{(1)}(\hbar\omega) \). According to (102), we have

\[
\phi_R^{(1)}(\hbar\omega) = \frac{\partial}{\partial E_{cl}} \arctan \left( \frac{2B}{A} \frac{1 - \frac{B^2}{A^2}}{1 - \frac{B^2}{A^2}} \right) \bigg|_{E_i=0} \cdot \hbar\omega
\]

\[
= \frac{2}{B(\xi_0)} \frac{\partial}{\partial E_{cl}} \left( \frac{C}{A} \left( \xi_0 + \frac{E_{cl}}{mga} \right) \right) \bigg|_{E_i=0}
\]

\[
= \frac{\pi}{mga} \left( \xi_0 [Ai(\xi_0)]^2 - \left[ \frac{dAi}{d\xi}(\xi_0) \right]^2 \right) .
\]

(103)

where we have used the fact that \((B/A)(0)=0 \) [see Eq. (91)].

According to Eq. (91) we have

\[
\frac{d}{dE_{cl}} \left( \frac{B}{A} \right) \bigg|_{E_i=0} = \frac{1}{B(\xi_0)} \frac{\partial}{\partial E_{cl}} \left( \frac{C}{A} \left( \xi_0 + \frac{E_{cl}}{mga} \right) \right) \bigg|_{E_i=0}
\]

\[
= \frac{\pi}{mga} \left( \xi_0 [Ai(\xi_0)]^2 - \left[ \frac{dAi}{d\xi}(\xi_0) \right]^2 \right) .
\]

(104)

Since the argument \( \xi_0 \) is a negative number of large absolute value, we can use the asymptotic form of the functions \( Ai(\xi) \) and \( dAi/d\xi \). We already know the former [see Eq. (95)]. The later can be found in Ref. 49. It reads (up to its leading term)

\[
\frac{dAi}{d\xi} \sim \frac{1}{\sqrt{\pi}} |\xi|^{1/4} \sin \left( \frac{2}{3} |\xi|^{3/2} - \frac{\pi}{4} \right) .
\]

(105)
Using (95) and (105) into (104) and recalling that $\xi_0 < 0$, we find
\[
\frac{\partial}{\partial E_{cl}} \left( \frac{B}{A} \right) \bigg|_{E_{cl} = 0} = -\frac{1}{\hbar} \sqrt{2 \left( \frac{b - y_0}{g} \right)},
\]
so that (103) is simply given by
\[
\varphi^{(1)}_R(n\hbar \omega) = -2n\hbar \sqrt{\frac{2(b - y_0)}{g}}.
\]  
We now write $T = 2 \sqrt{\frac{2(b - y_0)}{g}}$ and substitute (107) into (101) to get the explicit expression of $\psi^{\alpha}_{k, \text{ref}}(y)$,
\[
\psi^{\alpha}_{k, \text{ref}}(y) \sim \frac{1}{2\sqrt{n!}} |\xi|^{-1/4} e^{-i((2/3)|\xi|^{3/2} - \pi/4)} e^{-in\omega T}.
\]  
The reflected part of the eigenstate $\Psi(x, y, q)$ [see Eq. (93)] reads
\[
\Psi_{E, k, \text{ref}}(x, y, q) = C_{E, k} \frac{e^{ikx}}{2\sqrt{\pi}} |\xi|^{-1/4} e^{-i((2/3)|\xi|^{3/2} - \pi/4)} \sum_{n=-j}^j e^{in(q - \omega T)} \sqrt{2\pi N}.
\]  
According to Sec. V [see Eq. (64) and the discussion below that equation], we recognize the shifted clock state $\psi_0(q - \omega T)$ as one of the factors in the reflected wave-function, as expected. We can thus finally conclude that the time of flight of a quantum particle in a uniform gravitational field, in the context of NCQM, is given by
\[
T = 2 \sqrt{\frac{2(b - y_0)}{g}}.
\]  
In order to understand the physical meaning of the above result, we remark that Eq. (110) is equal to the corresponding expression of the ordinary quantum mechanics. More than this, it is equal to the classical result, thus showing that the weak equivalence principle extends to the case of quantum mechanics with space-space noncommutativity of the canonical type.

To finish, it is important to clarify a difference between the use we did of Peres approach in computing the time of flight of a particle in a uniform gravitational field (in NCQM) and the use of it by Davies (in ordinary quantum mechanics). First of all, it is important to bear in mind that the central idea of Peres is to couple a particle to a quantum system (that plays the role of a quantum rotor), which will work as a quantum clock. When this coupling is considered small, it is such that the effect of the particle-clock interaction is essentially only to cause a displacement of the (initial) position of the clock pointer, i.e., to change the initial state of the clock. Thus, at the end of the interaction, the clock will record the time of flight of the particle, which is simply the time interval during which the particle interacted with the clock-rotor.

In order that one can “watch” the final position of the clock pointer, and thus get the time of flight of the particle, it is necessary that a measurement be realized. This, by its turn, must be done in the “distant future,” when the particle-clock interaction can be considered totally negligible, which is one of the key points of Peres approach. By analyzing the scattered wave-function, we have thus shown that the time of flight is codified in its phase shift [see Eqs. (107)–(110)]. This result is totally in accordance with the spirit of that proven by Peres in the case of a free particle.

In the approach by Davies for computing the time of flight of a particle in a uniform gravitational field, the Hamiltonian of the system does not contain the coupling term between the
particle and the clock. In this case, the solution of the scattering problem does not “see” the clock, but only the gravitational barrier. As such, what Davies determined was not the phase shift of the wave-function due to the interaction between the particle (under the action of gravity) and the clock barrier. In fact, what he has done was to calculate the derivative (with respect to the total energy) of the phase difference induced by the action solely of gravity on the particle wave-function. This derivative was calculated at the point where the clock detector should have been placed (it should be noted that this derivative is position dependent). Nevertheless, it follows that the final result for the time of flight obtained by Davies turns out to be the same one that would be obtained by Peres approach, described above. Hence, in Davies approach the clock is introduced \textit{ad hoc} for the computation of the time of flight, through the prescription that the derivative of the phase difference of the particle wave-function has to be calculated at the point which corresponds to the position where the clock detector should be placed. We remark that in Peres approach, which is based on the calculation of the phase shift of the wave-function, there is no need of dealing with position dependent phases.

**VII. CONCLUDING REMARKS**

We have studied the motion of a particle under the action of a uniform gravitational field in NCQM. Assuming noncommutativity only on configuration space, we have \textit{exactly} solved the noncommutative Schrödinger equation and determined the energy eigenvalues after we carefully studied the self-adjointness of the operator involved as well as determined its self-adjoint extensions. We thus concluded that the usual boundary condition associated with the reflecting mirror at the bottom of the gravitational well is among those permitted by the theory of self-adjoint extensions when applied to the original operator we started from. As in ordinary quantum mechanics, in the noncommutative case the solution is given by the Airy function and the energy eigenvalues are expressed in terms of the zeros of the Airy function $Ai$. Obtaining the energy spectrum is especially important, since from the data of the gravitational quantum well experiment with freely falling neutrons (the GRANIT experiment),\textsuperscript{18,19} we could then set an upper-bound on the value of the spatial noncommutative parameter $\theta$. This experimental result can be improved in the future, when more accurate experimental data will be available.\textsuperscript{19} We note that the works that considered the noncommutative gravitational quantum well have not established an upper-bound on $\theta$, but rather on the momentum-momentum noncommutativity parameter\textsuperscript{20–23} or on the time-space component of the noncommutative matrix, $\theta_{01,2}$.\textsuperscript{22}

Another issue we have addressed in this paper was related to the question of the (weak) equivalence principle, its validity in NCQM. We were interested in investigating the status of the equivalence principle through a kind of (ballistic) Galileo experiment, associated with delocalized, energy eigenstates. For that, we have used a quantum clock model, due to Peres,\textsuperscript{32} in order that the time of flight of the particle could be measured. We remark that although we were inspired by Davies\textsuperscript{31} in asking for a possible violation of the equivalence principle by studying the time of flight of a particle subjected to a uniform gravitational field, we have performed a conventional stationary state analysis of a scattering problem, instead of just naively applying the formula used by Davies in the ordinary case.\textsuperscript{31} Instead of that, we closely followed the original approach due to Peres\textsuperscript{32} of applying a quantum clock in the measurement of the time of flight of a quantum particle (see also Ref. 58). It resulted that the time of flight is the same as in quantum mechanics, which in turn is identical to the classical result, when the measurement is made far from the turning point. This result can be interpreted as an extension of the equivalence principle to the realm of NCQM.

In order to address until to what extent the (weak) equivalence principle holds in NCQM (and analogously in ordinary quantum mechanics), further studies are important, such as, for instance, the investigation of how matter and antimatter, in a noncommutative background, behave under the action of a gravitational field. Although it has been shown that in NCQFTs the CPT symmetry is still preserved,\textsuperscript{60} but with individual violations of the C, P, and T symmetries, such a symmetry might be broken at a more fundamental level, when not only the matter fields are quantized but also the gravitational field itself. In fact, it has been suggested that quantum gravity effects might
lead to violations of the CPT symmetry. Finally, we recall that as the noncommutativity of space-time might be a signature of quantum gravity, experiments involving an interplay between gravity and quantum mechanics are welcome, since even at low-energies, but with enough level of accuracy, they might display information about the physics whose origin is in a more fundamental level (see Ref. 62).

It would be interesting to experimentally investigate the equivalence principle by measuring the time of flight of quantum particles using a quantum clock. Of course that one has to deal with the intrinsic uncertainty of using a quantum clock, as even at low energy (weak coupling between the clock and the system of interest), a good measurement of the time of flight is subjected to a lower limit on the time resolution of the clock. In this direction, we quote the work of Alonso et al., which has shown, in the case of a free quantum particle, that it is possible to gain an improvement in the measurement of time of flight if the quantum clock does not work continuously but rather by means of pulsed couplings.

ACKNOWLEDGMENTS

The authors would like to dedicate this work in memoriam to Ivens Carneiro, a great friend and admirable physicist, who prematurely passed away. The authors would like to thank the referee for his/her comments which led to improvements in the paper. The authors are grateful to Professor D. A. T. Vanzella for a helpful discussion. K.H.C.-B. acknowledges Professor D. A. T. Vanzella for encouragement, and also Grupo de Física Teórica, do Departamento de Física e Informática, do Instituto de Física de São Carlos, for hospitality. A.G.M. thanks the Universidade de Brasília for the hospitality during part of this work.


41. A subset $D$ of $H$ is dense (in the topology of $H$) if for every $\phi \in H$ there exists a sequence $\{\phi_j\}$, $\phi_j \in D$, such that $\phi_j \to \phi$ as $j \to \infty$. Besides, we have $D(H) = H$, where $D(H)$ is the closure of $D(H)$.
42. Let $T$ be a densely defined operator and $D(T^*) = \{\psi \in H : \exists \eta \in H : \langle T\eta, \psi \rangle = \langle \phi, \eta \rangle, \forall \phi \in D(T)\}$. We define the adjoint $T^*$ of $T$ as $T^* \phi = \eta$, $\forall \psi \in D(T^*)$.
43. The $\eta$th weak derivative of the integrable function $u$ is the integral function $v$, such that, for all $\phi \in C_0^\infty(a, b)$, we have $\int_a^b \eta^j \phi(t) \, dt = \int_0^\eta \frac{d^j}{dt^j} \phi(t) \, dt$.
47. An operator $T$ is called closed if and only if its graph, defined as $G(T) = \{(\phi, \psi) \in H \times H : \psi = T\phi\}$, is closed. If an operator $T$ has a closed extension, then it is called closable. The closure of a closable operator $T$ is the operator $\overline{T}$ defined by $G(\overline{T}) = \overline{G(T)}$. It turns out that $\overline{T}$ is the smallest closed extension of $T$.