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On quantum integrability of the Landau–Lifshitz model

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We investigate the quantum integrability of the Landau–Lifshitz (LL) model and solve the long-standing problem of finding the local quantum Hamiltonian for the arbitrary $n$-particle sector. The particular difficulty of the LL model quantization, which arises due to the ill-defined operator product, is dealt with by simultaneously regularizing the operator product and constructing the self-adjoint extensions of a very particular structure. The diagonalizibility difficulties of the Hamiltonian of the LL model, due to the highly singular nature of the quantum-mechanical Hamiltonian, are also resolved in our method for the arbitrary $n$-particle sector. We explicitly demonstrate the consistency of our construction with the quantum inverse scattering method due to Sklyanin [Lett. Math. Phys. \textbf{15}, 357 (1988)] and give a prescription to systematically construct the general solution, which explains and generalizes the puzzling results of Sklyanin for the particular two-particle sector case. Moreover, we demonstrate the S-matrix factorization and show that it is a consequence of the discontinuity conditions on the functions involved in the construction of the self-adjoint extensions. © 2009 American Institute of Physics. [doi:10.1063/1.3231789]

\section*{I. INTRODUCTION}

The Landau–Lifshitz (LL) model has been the subject of great interest in low-dimensional condensed matter physics as a model describing continuous classical magnets (for a review, see Refs. 1 and 2). In recent years, there was a surge of interest toward the LL model in relation with the gauge/string duality, where the LL model appeared on both sides of the correspondence.\textsuperscript{3–9} In both one-dimensional magnetism in condensed matter physics and in the context of the gauge/string duality it has become clear that the integrability plays the crucial role (for a review, see Refs. 10–13), allowing to construct the exact solutions and revealing the rich structure of the spectrum. Despite many years of investigation, only the classical theory of the LL model has been thoroughly investigated. The classical solitonic solutions have been found and discussed extensively in Refs. 14–19. More complete classical analysis became possible after the classical integrability was established for the isotropic case first in Refs. 14 and 20 and for the general anisotropic case in Refs. 19 and 21. The action-angle variables were constructed in Ref. 15, and in Ref. 22 the classical equivalence between the LL and the nonlinear Schrödinger (NLS) models was established, relating the flat currents of the corresponding models by a gauge transformation. The quasiclassical spectrum was analyzed in Ref. 23 and subsequently in Ref. 16.

In contrast, the development of the quantum theory of the LL model was affected by a number of missed subtleties and nuances, which, as a consequence, led to the wrong quantization procedure and incorrect results.\textsuperscript{24,25} Let us remind the general procedure of quantizing continuous integrable models. Apart from a few specific models, for example, the NLS and the fermionic Thirring models, which can be quantized directly in the continuous case by means of the inverse

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scattering method as well as by coordinate Bethe ansatz, it is a standard procedure to consider first
the lattice version of the continuous theory. This is done to deal with the ultraviolet divergences
and regularize ill-defined operator product at the same point. The quantum Hamiltonian, and other
conserved charges, can be found then by using the well-defined trace identities. Although there are
many lattice models corresponding to the same continuous theory,\textsuperscript{26–31} the requirement of in-
tegrability restricts (but not eliminates completely) the choice of the corresponding lattice model. A
systematic method that in principle is applicable for any integrable continuous model was outlined
in Ref. 26 (see also Ref. 31). In Ref. 26 this program has been successfully implemented for the
NLS model and the sine-Gordon models. The difficulties with this procedure were already em-
phasized in the original paper. Even for simplest NLS model, the construction turned out to be
quite nontrivial, and the resulting quantum Hamiltonian had a form describing interaction of eight
nearest neighbors. For the sine-Gordon model, and, in general, for other continuous models, the
quantum Hamiltonian is nonlocal, namely, it describes the interaction which depends on all lattice
sites. An alternative method suggested in Refs. 27 and 28 states the existence of local quantum
Hamiltonians for continuous integrable models, but its practical construction, based on the repre-
sentations of the Sklyanin algebras,\textsuperscript{32,33} is a complex and, in general, unresolved problem (see also
Ref. 34). Although for the specific LL model, its lattice version is known, and can be obtained
from the XYZ spin chain, for other more complex continuous integrable models, the resulting
lattice regularized quantum Hamiltonian will, in general, be of a quite complex form, if con-
structed following the procedure of Ref. 26. The nonlocality of the Hamiltonian may be a serious
barrier to deal with, if one is interested in other subtle properties of the system. Thus, the quant-
ization of the continuous integrable systems is highly desirable to carry out directly, without first
passing to the lattice version.

In Ref. 24 such program was first initiated, revealing a number of interesting questions and
nuances, associated with quantization of the continuous integrable systems. There are several
important points that should be addressed. To begin with, the usual method of constructing the
quantum Hamiltonian and the other conserved charges, which works in lattice models due to the
well-defined expressions and a few continuous models (the NLS model), does not work for the LL
and, in general, for the majority of continuous models. The formal usage of the trace identities
analogous to the ones of the NLS model or the lattice models, leads to wrong results.\textsuperscript{25} There is,
essentially, no effective or systematic method of constructing the local conserved charges in
continuous integrable models, as the quantum corrections modify the formal ill-defined expres-
sions, which follow from the formal trace identities. In Sklyanin’s original paper only the action of
the two-particle quantum Hamiltonian was found, which involved some guess work and consist-
tency with the classical and quasiclassical cases. The constructions also required an unusual space
of quantum states in the quantum-mechanical picture and very specific continuity properties of the
functions involved. The higher \( n \)-particle sector quantum Hamiltonian, its action, and the quantum
states have been unknown until now. Surprisingly, this matter has not been investigated in detail,
despite its obvious importance.

There are other interesting subtleties arising in the quantization of the LL model. Namely, in
the anisotropic case the standard passing from the classical to the quantum transfer matrix does not
work. Instead, the \( R \)-matrix, as well as the monodromy matrix, is essentially guessed to satisfy the
Yang–Baxter and bilinear relations. The construction of the monodromy matrix requires an addi-
tional spin operator, which, as a result, changes the algebra of the spin operators, thus, giving rise
to the Sklyanin algebra.\textsuperscript{32,33} Although the existence of such algebras was known before for the
lattice systems, in the context of the continuous model its appearance is not very clear and remains
an open problem.

More importantly, it was realized in Ref. 24 that there are essentially two distinct classes of
the LL model, corresponding to the \( su(1,1) \) hyperboloid and \( su(2) \) sphere cases. As it was cor-
rectly pointed out by Sklyanin, and missed by others who attempted to quantize the LL model,
only in the \( su(1,1) \) case one may construct physically meaningful states. In the \( su(2) \) case, the
scalar product turns out to be not positively defined. Let us note that the statement above is for the
ferromagnetic case. It is possible to construct a positively defined scalar product for nonferromag-
nentic vacuum in the $su(2)$ case.\textsuperscript{35} It is well known, however, that in the case when the ferromagnetic vacuum does not exist, the algebraic Bethe ansatz is not applicable, and a more sophisticated construction, similar to the construction for the sinh-Gordon model, is needed.\textsuperscript{36,37} This problem will be considered separately.

In this paper we consider the isotropic $su(1,1)$ LL model and construct the desired quantum Hamiltonian for arbitrary $n$-particle sector, and its action on the states, which we also construct in detail. We recover the correct spectrum and consider in detail the continuity properties of the functions involved. For the specific case of the two-particle sector, our results exactly reproduce Sklyanin’s construction. The main point of our method is the regularization of the Hamiltonian directly in the continuous case. We achieve this by employing the split-point regularization, which effectively makes the Hamiltonian nonlocal, and constructing the space of quantum states, which requires a careful analysis of self-adjoint extensions in agreement with the scalar product. We emphasize that after removing the regularization, the Hamiltonian is local. As we show below, the quantum-mechanical Hamiltonian for the LL model yields a highly singular potential which contains second derivatives of the delta function. It is well known that in the space of functions with the usual scalar product, it is possible to construct the self-adjoint extensions only up to the first derivative of the delta function.\textsuperscript{38} More precisely, there exists a four-parameter solution in the space of function that contains the derivative of the delta-function potential, and no solution exists which contains the second derivative of the delta function. Thus, to circumvent this barrier, we propose to construct the self-adjoint extensions in the space of vectorlike states, which is a novel feature, not considered in the literature previously, to the best of our knowledge. This provides an explanation to the \textit{ad hoc} solution found by Sklyanin for the quantum-mechanical Hamiltonian for the two-particle sector. Although we consider here in detail only the LL model as the simplest illustrative example, the above-mentioned difficulties and subtleties of a similar to the LL model character will exhibit themselves also in other continuous integrable models. Thus, our method is general enough to be applicable to a wide range of continuous integrable models.

We also demonstrate in our method the S-matrix factorization, which is the underlying property of quantum integrability. It is worth noting here that we have already considered the three-particle S-matrix factorization for the LL model in our previous paper.\textsuperscript{39} There, we had shown that in the first nontrivial order of perturbation series, the three-particle S-matrix is indeed factorizable into a product of the two-particle S-matrices. Our work was based on earlier calculations of Ref. 40, where the LL model was considered from the field-theoretic point of view, and the exact two-particle S-matrix was found by summing up the bubble diagrams, surviving in the two-particle scattering process (see also Ref. 41 regarding the problems with diagonalization). We would like to emphasize the difference between the $su(1,1)$ LL model considered here and the LL model considered in Ref. 40. Only in the $su(1,1)$ case one can construct a consistent quantum theory with the ferromagnetic vacuum. It appears that the LL model considered in Ref. 40 actually corresponds to the $su(2)$ case. As we explained before, the construction of positive defined metric in the space of states corresponding to such a vacuum is mathematically challenging task and was considered in Ref. 35. The types of excitations in the $su(1,1)$ LL model and the case of Ref. 40 are also different. We do not see direct connection between the two models at this point, although this is an interesting problem to reconstruct the results and the spectrum of Ref. 40 following the inverse scattering method and the method we propose in this paper of regularization of the local conserved charges together with construction of the self-adjoint extensions. It is not surprising then that the $n$-particle S-matrix we find here for the $su(1,1)$ LL model (which is in complete agreement with the Sklyanin’s result) is different, albeit by a coefficient, from the S-matrix found by Ref. 40.

Our paper is organized as follows. In Sec. II, we briefly review the LL model in the context of the inverse scattering method. In Sec. III A, we illustrate our method on the simplest two-particle case, give the regularized continuous quantum Hamiltonian, construct the self-adjoint extensions, and derive, in complete agreement with, Ref. 24 the spectrum and the continuity properties of the functions involved. In Sec. III B, we consider the general $n$-particle case and show that the regularized quantum Hamiltonian used in the two-particle case is enough to construct the self-
adjoint extensions and the spectrum in this general case. In Sec. IV, we show the S-matrix factorization as the consequence of our construction. In Sec. V, we give a brief summary of our results and outline future problems.

II. THE LL MODEL: QUANTUM INVERSE SCATTERING METHOD

In this section we review, following Ref. 24, the main features of the LL model in the context of the inverse scattering method and discuss the arising difficulties and subtleties of this approach to quantization of the system.

The Hamiltonian for the anisotropic LL model has the following form:

\[
H = \frac{\epsilon}{2} \int \left[ - (\partial_x S, \partial_x S) + 4 \gamma^2 ((S^3)^2 - 1) \right],
\]

where the vector \( S = (S^1, S^2, S^3) \), and the scalar product is defined as \( (S, S) = (S^3)^2 - \epsilon (S^1)^2 - \epsilon (S^3)^2 = 1 \). The Poisson structure has the form

\[
\{S^0(x), S^\pm(y)\} = \pm i S^\pm(x) \delta(x-y),
\]

\[
\{S^-(x), S^+(y)\} = 2i \epsilon S^3(x) \delta(x-y),
\]

where \( S^\pm = S^1 \pm i S^2 \). Here \( \gamma \) is the anisotropy parameter, and the choice \( \epsilon = \pm 1 \) corresponds to the \( su(1,1) \) and \( su(2) \) cases correspondingly. [As we mentioned in Sec. I, in this article we will only consider the \( su(1,1) \) case, which, unlike the \( su(2) \) case, corresponds to the physically meaningful states in the ferromagnetic vacuum, with the particular choice of the representation for the states.] As explained in Ref. 24 the isotropic (\( \gamma = 0 \)) and anisotropic (\( \gamma \neq 0 \)) cases are essentially different and should be considered separately. In the former case, the conventional inverse scattering procedure goes through without any changes—the Yang–Baxter and bilinear equations are satisfied with the appropriate choice of the \( R \)-matrix. This is in contrast to the latter case, where the monodromy matrix and the spin operator algebra (Sklyanin algebra) have to be modified by hand for the intertwining relation (7) to have a solution,\(^{32,33}\) with the \( R \)-matrix corresponding to the XXZ model. The Sklyanin algebra naturally appears in lattice systems\(^{32}\) as a consistency condition of the intertwining relations (7). It is less clear, however, how to derive the Sklyanin algebra directly in the continuous case, and, in particular, in the LL model. We will consider here only the isotropic case.

The Poisson structure (2) is replaced by the commutation relations for the \( S \)-operators in the standard manner,

\[
[S^0(x), S^\pm(y)] = \pm S^\pm(x) \delta(x-y),
\]

\[
[S^-(x), S^+(y)] = 2i \epsilon S^3(x) \delta(x-y).
\]

The vacuum considered here corresponds to the \textit{ferromagnetic} case,

\[
S^0(x)|0\rangle = \epsilon |0\rangle,
\]

\[
S^-(x)|0\rangle = 0.
\]

The quantum \( \mathcal{L} \)-operator in the isotropic case takes the form

\[
\mathcal{L}(u,x) = \left( \begin{array}{c} S^0(x) - S^0(x) \\ 0 \\ S^-(x) - S^-(x) \end{array} \right),
\]

and the corresponding monodromy matrix is given by the expression
To construct the representations of

$$\Omega = P e^{\int_{-L}^{L} L(u(x)) dx} \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},$$

Here \( u \) is the spectral parameter and \( P \) signifies the path-ordered exponential. In this case the bilinear relation

$$(1) \quad R(u_1 - u_3) T(u_1) T(u_2) = T(u_2) T(u_1) R(u_1 - u_2)$$

is satisfied with the quantum \( R \)-matrix given by the following form:

$$R(u) = w_a(u) \sigma_a \otimes \sigma_a,$$

where the summation over the index \( a = 0, 1, 2, 3 \); \( w_0(u) = u - i/2 \); \( w_1(u) = w_2(u) = w_3(u) = -i/2 \). (We use the standard notation \( T = T \otimes 1 \) and \( T = 1 \otimes T \).)

To construct the representations of (3) in the ferromagnetic vacuum, one writes the vector in the form, analogous to other continuous integrable models, 31

$$|f_n\rangle = \int dx_1 \cdots dx_n f_n(x_1 \ldots x_n) S^*(x_1) \cdots S^*(x_n) |0\rangle,$$

where \( f_n(x_1 \ldots x_n) \) are continuous and decreasing sufficiently fast functions for the integral (9) to be well defined. A simple calculation shows 35 [see also Eq. (41) below] that the scalar product \( \langle g_m | f_n \rangle \) is positively defined only for the \( su(1, 1) \) case, while for the \( su(2) \) case, the matrix element is indefinite. Thus, only in the \( su(1, 1) \) case one is able to construct physically meaningful states in the ferromagnetic vacuum. Therefore, one has to take \( \varepsilon = 1 \) in (1)–(4). After passing to the infinite interval, the operators in the monodromy matrix (6) will satisfy the standard commutation relations, and together with the choice of the ferromagnetic vacuum, one can apply the well-known procedure of the algebraic Bethe ansatz to derive the spectrum and the eigenfunctions, which have the form (see Ref. 24, for complete details),

$$|u_1 \cdots u_N\rangle = B(u_1) \cdots B(u_N) |0\rangle.$$

In the classical and lattice models, the bilinear relation (7) guarantees the existence of the integrals of motion, which are obtained from the generating functional \( I(u) = \text{Tr}[T(u)] \). Here, however, one faces a difficulty, which is not present in the classical counterpart or the lattice version. Namely, in the classical case, one can simply decompose \( I(u) \) in the series,

$$I(u) = \sum_k I_k u^k,$$

to obtain the local integral of motion. For example, the classical Hamiltonian (1) can be shown to be one of the charges in the \( I_k \) series. In the quantum case, however, the difficulty is that the local charges, in particular, the Hamiltonian (1), contain operator product at the same point, thus making the local integrals of motion not well-defined quantities. Thus, the formally defined series (11) cannot be used to obtain the integrals of motion, and, in particular, the quantum Hamiltonian is not a priori known. Thus, construction of the local integrals of motion turns out to be a nontrivial problem in the continuous quantum theory. From the field theory point of view, this should correspond to the renormalization procedure, which to the best of our knowledge has not yet been performed for the LL model.

In Ref. 24 the quantum Hamiltonian was not found, and only the action of the (local) quantum-mechanical Hamiltonian on one- and two-particle sectors was presented. To formulate it, it was necessary to introduce new bosonic fields \( \Psi_n(x) \), corresponding to the \( n \)-particle clusters, so that \( \Psi_n(x) |0\rangle = 0 \), satisfying the following algebra:
One can then represent the $S$-operators as the following $n$-particle cluster decomposition,

\[
S^3(x) = s_0^3 + \sum_{n=1}^{\infty} s_n^3 \Psi_n^*(x) \Psi_n(x),
\]

\[
S^+(x) = s_0^+ \Psi_1^*(x) + \sum_{n=1}^{\infty} s_n^3 \Psi_{n+1}^*(x) \Psi_n(x),
\]

\[
S^-(x) = s_0^- \Psi_1(x) + \sum_{n=1}^{\infty} s_n^3 \Psi_{n+1}^*(x) \Psi_n(x),
\]

where $s_0^+=1$, $s_0^-=\sqrt{2}$, $s_n^+=n$, and $s_n^-=\sqrt{(n+1)n}$ $(n \geq 1)$. Using this cluster decomposition one can show that the two-particle eigenstate (10) has the form

\[
|u_1,u_2\rangle = -\frac{1}{2 \cos u_1 \cos u_2} \left[ \int dx e^{i(p_1+p_2)x} \Psi_2^*(x) + \int_{x_1>x_2} dx_1 dx_2 (c(p_1,p_2) e^{i(p_1 x_1+p_2 x_2)}
\]

\[+ \bar{c}(p_1,p_2) e^{i(p_1 x_2+p_2 x_1)} \Psi_1^*(x_1) \Psi_1^*(x_2) \right] |0\rangle,
\]

where

\[
c(p_1,p_2) = \frac{2(p_1-p_2) + ip_1 p_2}{2(p_1-p_2)}.
\]

Note, that in other known continuous models, solved by the coordinate Bethe ansatz, the first term in (14) is absent. This is the case, for example, for the bosonic NLS and fermionic massive Thirring models. This somewhat unusual feature of the LL model, which was also discussed in Ref. 39, will be given explanation in Sec. III when constructing the self-adjoint extensions.

As we mentioned earlier, the quantum field-theoretic Hamiltonian was not found in Ref. 24, but the action of the quantum-mechanical Hamiltonian on the two-particle sector was essentially guessed. To write it explicitly, it was necessary to introduce the space spanned by the states of the following type:

\[
|f\rangle = \begin{pmatrix} f_1(x) \\ f_2(x_1,x_2) \end{pmatrix} = \left[ \int dx f_1(x) \Psi_2^*(x) + \int_{x_1>x_2} dx_1 dx_2 f_2(x_1,x_2) \Psi_1^*(x_1) \Psi_1^*(x_2) \right] |0\rangle
\]

so that the function $f_2(x_1,x_2)$ is symmetric and smooth everywhere except on $x_1 \neq x_2$ line and $f_1(x) = f_2(x,x)$. The Hamiltonian action is defined as follows:

\[
H_2|f\rangle = \left( 2(\partial_{x_1} - \partial_{x_2}) f_2 |_{x_1=x_2} + \partial_{x_1}^2 f_2(x_1,x_2) \right) - \left( \partial_{x_1}^2 + \partial_{x_2}^2 \right) f_2(x_1,x_2).
\]

It is not difficult to check that the above action of the Hamiltonian $H_2$ is Hermitian with respect to the scalar product associated with the following norm:

\[
\|f\|^2 = \frac{1}{4} \int dx |f_1|^2 + \int_{x_1>x_2} dx_1 dx_2 |f_2(x_1,x_2)|^2.
\]

One can check, that the solution to (16)–(18) leads exactly to the two-particle state (14).
III. QUANTUM HAMILTONIAN AND SELF-ADJOINT EXTENSIONS

As we discussed in Sec. I, the construction of the local quantum Hamiltonian is a complicated task, in general. The standard procedure of putting the continuous theory on the lattice to regularize the ultraviolet divergences leads generally to nonlocal Hamiltonians, and only the existence of the local form can be proven, while construction in practice is a complicated and unresolved problem. On the other hand, only a few continuous integrable models (NLS, massive Thirring model) allow direct quantization of the system by coordinate Bethe ansatz, without using the inverse scattering method. The essential difference of the LL model from the NLS or Thirring models is the presence of more severe singularities in the quantum-mechanical Hamiltonian. Indeed, in the NLS model, the quantum-mechanical interaction is described by the $\delta(x)$ potential, while in the case of the LL model the interaction is highly singular and is proportional to $\partial_x \partial_y \delta(x-y)$. The standard procedure to deal correctly with such singular potentials is to construct self-adjoint extensions. Even though for the NLS and Thirring models the problem was solved without constructing self-adjoint extensions, in general, this is not correct.

The construction of self-adjoint extensions for the LL model is, however, immediately bounded by the following fact. It is known that in the space of functions with the usual scalar product, it is possible to construct the self-adjoint extensions only up to the first derivative of the delta function, namely, there exists a four-parameter extension in the space of function that contains the derivative of the delta-function potential. Since for the LL model the interaction is of the second order derivative of the delta function, the above statement means that one has to construct a different scalar product in the new space of function. In Sec. III A we will present this construction in detail first for the two-particle case, before generalizing our analysis for the $n$-particle case. We will derive the Sklyanin’s result (16)–(18) and find its $n$-particle extensions. To do this, we will also propose the regularized continuous quantum Hamiltonian that will be checked to give correct results for any $n$-particle sector.

A. Two-particle sector

It is easier to demonstrate the idea of our method on the two-particle case, and in the more general case, considered in Sec. III B, the complications are only of the technical character. Let us begin by writing down the regularized quantum Hamiltonian, corresponding to (1). It is not difficult to see that the direct application of the Hamiltonian (1) to the two-particle state (9),

$$|f_2\rangle = \int dx dy f_2(x,y)S^+(x)S^+(y)|0\rangle,$$

leads to undefined singular expressions of the type $\partial_x^2 \delta(x)|_{x=0}$. The idea is to regularize the continuous Hamiltonian by the split-point method. Namely, we take our quantum Hamiltonian to be of the form

$$H_Q = \lim_{\epsilon \to 0} H_{\epsilon},$$

where

$$H_{\epsilon} = \frac{1}{2} \int du dv F_\epsilon(u,v)\left[ -\partial_u S^2 \partial_v S^3 + \partial_u S^3 \partial_v S^2 + \partial_u \partial_v (S^3(u)\delta(u-v)) - \partial_u \partial_v \delta(u-v) \right].$$

Here the function $F(u,v)$ is any smooth function, depending on some parameter $\epsilon$, so that...
We note here that in the more general case for the $n$-particle sector, these matching conditions, resulting after the action of the quantum Hamiltonian (20) on the $n$-particle state, will remain the same, with the obvious change $x \rightarrow x_i$ and $y \rightarrow y_j$. After some straightforward algebra, we can derive the action of the quantum Hamiltonian (20) on the $n$-particle state,

$$H_Q|\phi_n\rangle = -\int d\bar{x} (\Delta f(\bar{x})) \prod_{i=1}^{n} S^*(x_i)|0\rangle + \sum_{i>j} \int d\bar{x} \{(\partial_{\bar{x}} f(\bar{x}) - \partial_{\bar{x}} f(\bar{x})) \}^{\{x=\bar{x}\}}_i^{\{y=\bar{y}\}} + \partial_{\bar{x}} f(\bar{x}) \prod_{i=1}^{n} S^*(x_i)|0\rangle. \tag{27}$$

From here we immediately derive the general matching conditions,

$$-(\partial_{x} f) \}^{\{x=\bar{x}\}}_i^{\{y=\bar{y}\}} = \partial_{\bar{x}} f(\bar{x}) \}^{\{x=\bar{x}\}}_i^{\{y=\bar{y}\}} \quad \forall \ i > j. \tag{28}$$

This is, in fact, the reason for the $S$-matrix factorization, as we will see below. We also emphasize that the quantum Hamiltonian (20) does not acquire any further corrections in the higher $n$-particle sectors.

We will show now that the matching condition (26) together with the equation following from (25)
where \( E_2 \) is the energy of the two-particle state, recovers the Sklyanin’s solution (14). We will first construct the space on which the quantum-mechanical Hamiltonian acts.

Let us consider a space \( V \) generated by the vectors of the form

\[
\Psi = \begin{pmatrix} f_1(x) \\ f_2(x,y) \end{pmatrix},
\]

where the function \( f_1(x) \) is determined by \( f_2(x,y) \) and, possibly, its derivatives at \( x \to y \). The actual form of \( f_1(x) \) will be fixed later. For a given non-negative number \( \alpha \), we define a scalar product on \( V \) as follows:

\[
\langle \Phi | \Psi \rangle = \alpha \int_{-\infty}^{\infty} g_1^*(x)f_1(x)dx + \int \int_{x \neq y} g_2^*(x,y)f_2(x,y)dxdy,
\]

where

\[
\Psi = \begin{pmatrix} f_1(x) \\ f_2(x,y) \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} g_1(x) \\ g_2(x,y) \end{pmatrix}.
\]

We require the function \( f_1(x) \) to belong to \( \mathcal{L}^2(\mathbb{R}, dx) \) and \( f_2(x) \) to \( \mathcal{L}^2(\mathbb{R}^2/\{x=y\}, dx dy) \). Further conditions on \( f_1(x) \) and \( f_2(x,y) \) will be imposed later.

Now we define the operator, \( \hat{H} \), on \( V \) with the following properties.

(i) Acting on \( f_2(x) \) it is simply the Laplacian \( -\Delta = -\partial_x^2 - \partial_y^2 \) everywhere in \( \mathbb{R}^2/\{x=y\} \), i.e.,

\[
\hat{H}\Psi = \begin{pmatrix} \hat{h}f_1(x) \\ -\Delta f_2(x,y) \end{pmatrix}
\]

with some operator \( \hat{h} \) to be determined later.

(ii) It is Hermitian with respect to the scalar product (31), i.e.,

\[
\langle \Phi | \hat{H} | \Psi \rangle = \langle \hat{H}\Phi | \Psi \rangle.
\]

Using

\[
g^*(x,y)\Delta f(x,y) = \partial_i(g^*(x,y)\partial_i f(x,y)) - \partial_i(\partial_j g(x,y)) f(x,y) + (\Delta g(x,y)) f(x,y),
\]

where \( i = \{x,y\} \), and not assuming continuity neither functions nor their derivatives at \( x=y \), we have

\[
\int \int_{x \neq y} g_2^*(x,y)\Delta f_2(x,y)dxdy = \int \int_{x \neq y} (\Delta g_2(x,y))^*f_2(x,y)dxdy - \int_{-\infty}^{\infty} dy \int \int_{x \neq y} g_2^*(x,y)\partial_y f_2(x,y)
\]

\[
- \int_{-\infty}^{\infty} dx \int \int_{x \neq y} g_2^*(x,y)\partial_x f_2(x,y)
\]

Now, to be more specific, we are going to impose some conditions on \( f_2(x,y) \). We require it to be continuous at \( x=y \). (The other possibilities are also interesting but we consider the one that is relevant for our problem.) In this case Eq. (35) simplifies to become
\[
\int \int_{x+y} g_2(x,y) \Delta f_2(x,y) dx dy = \int \int_{x+y} (\Delta g_2(x,y))^\ast f_2(x,y) dx dy + \int_{-\infty}^{\infty} dx (g_2(x,x)[\partial_x f_2(x,y) - \partial_y f_2(x,y)])_{y=x-\varepsilon} - [(\partial_x g_2(x,y))^\ast - (\partial_y g_2(x,y))^\ast]_{y=x-\varepsilon} f_2(x,x) \). 
\]

(36)

It is obvious that the first line in (36) has needed for Hermiticity form, while the second and third lines should be compensated by \( \hat{h} \) [see Eq. (33)] in such a way that the full \( \hat{H} \) would become Hermitian. Before proceeding, we must fix the relation between \( f_1(x) \) and \( f_2(x,y) \). Looking at the one-dimensional integral in (34), one sees that a natural choice is \( f_1(x) = f_2(x,x) \) [which is possible after we required continuity of \( f_2(x,y) \)]. Now it is not difficult to see that the following form of \( \hat{h} \),

\[
\hat{h} f_1(x) := \frac{1}{\alpha} [\partial_x f_2(x,y) - \partial_y f_2(x,y)]_{y=x-\varepsilon} + \hat{h}_1 f_1(x),
\]

(37)

where \( \hat{h}_1 \) is any Hermitian in \( L^2(\mathbb{R}, dx) \), does the job.

Thus, we see that the requirement for \( \hat{H} \) to be Hermitian (or rather symmetric) with respect to the scalar product (31) does not fix the Hamiltonian completely even after we had chosen some conditions on the components of \( |\Psi\rangle \). But, in fact, we still have to use one consistency condition: the image of \( |\Psi\rangle \) under the action of \( \hat{H} \) should belong to the same class of vectors, namely, the first component should be related to the second one,

\[
\hat{h} f_1(x) := \frac{1}{\alpha} [\partial_x f_2(x,y) - \partial_y f_2(x,y)]_{y=x-\varepsilon} + \hat{h}_1 f_1(x) = - \Delta f_2(x,y)|_{x=y}.
\]

(38)

This put some constraints on the form of \( \hat{h}_1 \) as well as imposes some conditions on the behavior of \( f_2(x,y) \) at \( x=y \).

For the LL model, it is not difficult to show from (19) that the coefficient in the scalar product (31) \( \alpha = 1/2 \), and using Eq. (26) we find that \( \hat{h}_1 = -\partial_x^2 \). Thus, we obtain from (33)

\[
H_2[f] = \begin{pmatrix}
2(\partial_x^2 - \partial_y^2) f_{2(x_1,x_2)} |_{y=x-\varepsilon} & -\partial_x^2 f_2(x,x) \\
(\partial_x^2 + \partial_y^2) f_{2(x_1,x_2)} & \end{pmatrix}
\]

(39)

This is exactly the formula (17) guessed by Sklyanin.

B. \( n \)-particle sector

Let us consider a space of vectors of the form

\[
|f_n\rangle = \int d^n \tilde{\phi}(\tilde{x}) S^*(x_1) \cdots S^*(x_n) |0\rangle.
\]

(40)

Then it is not difficult to calculate a “scalar product” in this space. Let \( \{X_m\} \) be a partition of the set \( \{x_i\} \),

\[
\bigcup_{m=1}^{M_p} X_m = \{x_i\} \quad \text{and} \quad X_m \cap X_n = \delta_{mn} Y_m,
\]

and let \( t_m \) be a “collective” coordinate for all \( x_i \in X_m \). Then the resulting “scalar product” is
\begin{equation}
\langle g_n f_n \rangle = \sum_{\text{partitions}} e^{-M_P} C_P \int d^{M_P} g^n(\vec{x}) f(\vec{x})|_{(x_i = x_m)} \varphi_m.
\end{equation}

Here \( C_P \) are some combinatorial factors, which are positive. From here it is obvious that this "scalar product" will be an actual scalar product only for the noncompact case of \( su(1,1) \), where \( \epsilon = 1 \). From now on we concentrate on this case postponing consideration of the \( su(2) \) model to future work. It is instructive to write down \( n = 2 \) and \( n = 3 \) cases explicitly,

\begin{align}
n = 2: \langle g_2 f_2 \rangle &= 4 \left[ 2 \int dx dy g^2(x,y) f(x,y) + \int dx g^2(x,x) f(x,x) \right], \quad \text{(42)} \\
n = 3: \langle g_3 f_3 \rangle &= 8 \left[ 6 \int dx dy dz g^3(x,y,z) f(x,y,z) + (3 \cdot 3) \int dx dy g^3(x,y,z) f(x,y,z) + \int dx g^3(x,x,x) f(x,x,x) \right]. \quad \text{(43)}
\end{align}

While (42) is exactly the scalar product used in Ref. 24 [see also Eq. (31)], formula (43) is presented here to explicitly demonstrate the first nontrivial case.

There are a couple of useful interpretations of (41). The first one is along the lines of Ref. 24. Namely, one can think of (40) as a vector function with (41) as a natural scalar product. Even though this is a useful interpretation, mostly because this nicely fits into the cluster picture of Ref. 24, the more mathematically rigorous one is as follows. Namely, it is a scalar product in \( L^p(\mathbb{R}) \), where the completeness is defined with the help of the Riemann–Stieltjes integral, rather than just the Riemann one. Here, \( \mu \) is the "measure" for this integral, which formally could be written as

\begin{equation}
\int f(\vec{x}) d^p \mu(\vec{x}),
\end{equation}

where \( \mu \) now is not required to be a smooth function. The only requirement is that \( f \) and \( \mu \) do not have discontinuity at the same points, which is the case—our function is continuous. Moreover, that is why it is that hard to define the self-adjoint operator—one needs to be extremely careful at the points, where the measure is discontinuous. The measure \( \mu \) is very easy to read from the scalar product. In particular, for the case \( n = 2 \), Eq. (42), we have (formally)

\begin{equation}
\mu(x,y) = 1/2xy - x \theta(x-y) - y \theta(y-x)
\end{equation}

Regardless of the interpretation, the strategy in defining a self-adjoint operator is the same: because the continuity is a property only of the function itself but not of its derivatives, there will be surface terms that will compete with lower dimensional integrals. They should be accurately taken into account. That is exactly what we do below (and what we saw above for the \( n = 2 \) case).

Then we proceed exactly as we did in \( n = 2 \) case: we will construct a self-adjoint extension of the Hamiltonian, which in the "bulk," i.e., everywhere except \( x_i = x_j \) for all possible \( i \) and \( j \), reduces to a Laplacian,

\begin{equation}
-\Delta = - \sum_{i=1}^{N} \partial_i^2.
\end{equation}

As usual, this will amount to imposing some sewing conditions on derivatives of \( f(\vec{x}) \) at \( x_i = x_j \). Let us start with the integration of the highest dimensionality in (41), i.e., when all the clusters contain just one coordinate, \( X_i = x_i \). Performing integration by parts and taking into account surface terms, we have
\[
\int d^N g^*(\tilde{x}) \hat{\Delta}_i f(\tilde{x}) = \int d^N g^*(\hat{\Delta}_i g)^*(\tilde{x}) f(\tilde{x}) + \sum_{j \neq i} \int \prod_{k \neq i} dx_k [g^*(\tilde{x}) \partial_j f(\tilde{x}) - (\partial_j g)^*(\tilde{x}) f(\tilde{x})]_{x_k=x_{j-i}}^{x_i=x_{j-i}} \tag{47}
\]

Summing over \(i\), we have the result for the full Laplacian. It is clear that the action of the Hamiltonian on \((n-1)\)-components of the aforementioned vector function, i.e., when only two of the coordinates are equal, should exactly compensate the unwanted term,

\[
\sum_i \sum_{j \neq i} \int \prod_{k \neq i} dx_k [(\partial_j g)^*(\tilde{x}) f(\tilde{x})]_{x_k=x_{j-i}}^{x_i=x_{j-i}} \tag{48}
\]

and simultaneously reproduce the term

\[
- \sum_i \sum_{j \neq i} \int \prod_{k \neq i} dx_k [g^*(\tilde{x}) \partial_j f(\tilde{x})]_{x_k=x_{j-i}}^{x_i=x_{j-i}} \tag{49}
\]

What is the Hamiltonian acting on \(f_{ij} = f(\tilde{x})\bigg|_{x_i=x_j}\) that does this? The answer is obvious, if we recall that the function \(f\) itself is continuous. Then we can define

\[
\hat{h}_{ij} f_{ij} = -\alpha (\partial_j f(\tilde{x}) - \partial_j f(\tilde{x}))_{x_i=x_{j-i}}^{x_i=x_{j-i}} + \overline{h}_{ij} \tag{50}
\]

Here \(\alpha\) is a combinatorial coefficient expressed in terms of \(C_p\) from (41) [in fact, it is not hard to see that it is always the same as for \(n=2\): \(\alpha=2\), e.g., it is explicitly seen for \(n=3\) from (43): \(\alpha = \frac{6}{3} = 2\)]; \(\overline{h}_{ij}\) is some operator in \((n-1)\)-dimensional space, such that its non-Hermiticity is on the next, \((n-2)\)-dimensional, level. In a moment we will see that it is nothing but a Laplacian acting on \(f_{ij}\). It is also easy to understand why there are two terms in (50) instead of the one, as one would naively expect from (49): the term at \(x_i=x_j\) will arise twice in the sum—the first time as

\[
[g^*(\tilde{x}) \partial_j f(\tilde{x})]_{x_i=x_{j-i}}^{x_i=x_{j-i}} \tag{51}
\]

and the second time as

\[
[g^*(\tilde{x}) \partial_j f(\tilde{x})]_{x_i=x_{j-i}}^{x_i=x_{j-i}} \tag{52}
\]

The residual freedom in the definition of \(\hat{h}_{ij}\) coming from yet undefined \(\overline{h}_{ij}\) cannot be fixed only by requiring Hermiticity, \(\hat{h}_{ij}\) (and its analogs in lower dimensions), remain undetermined. It is removed by requiring that it should agree with the matching conditions that follow from the explicit action of the quantum Hamiltonian on the state (40). As we had already mentioned in Sec. II, the matching conditions, following from the action of the quantum Hamiltonian (20) on the \(n\)-particle state (40), are, essentially, the same as in the two-particle sector case (26). Below we show that providing the matching conditions (28),

\[-(\partial_j f(\tilde{x}) - \partial_j f(\tilde{x}))_{x_i=x_{j-i}}^{x_i=x_{j-i}} = \partial_j \partial_j f(\tilde{x})\big|_{x_i=x_{j-i}} \forall i > j, \tag{53}\]

then \(\hat{h}_{ij} f_{ij}\) is, in fact, equal to \(\Delta f(\tilde{x})\big|_{x_i=x_j}\) if \(\overline{h}_{ij}\) equals to \((n-1)\)-dimensional Laplacian.

Let us find how a Laplacian in a lower dimensional space is related to the one in \(n\)-dimensional evaluated when some coordinates coincide. For the future use we consider more general case when there is one cluster but not necessary of the size two, \(\{x_{k_1}, \ldots, x_k\}\). As above, let \(i\) be a collective coordinate for this cluster and \(i\) runs over the rest of the coordinates. Then the lower dimensional Laplacian is

\[
\int d^{N-i} g^*(\tilde{x}) \hat{\Delta}_i f(\tilde{x}) = \int d^{N-i} g^*(\hat{\Delta}_i g)^*(\tilde{x}) f(\tilde{x}) + \sum_{j \neq i} \int \prod_{k \neq i} dx_k [g^*(\tilde{x}) \partial_j f(\tilde{x}) - (\partial_j g)^*(\tilde{x}) f(\tilde{x})]_{x_k=x_{j-i}}^{x_i=x_{j-i}} \tag{54}
\]
the self-adjoint extensions and their continuity properties. Thus, having the exact expressions for direct fashion, following the results of the previous sections, where we have explicitly constructed $S$-matrix factorization. In fact, one can proceed in the same fashion as for the simpler NLS model of the $n$-particle case, which, of course, should be the case as Hamiltonian should take continuous function to continuous function.

Because the non-Hermitian part on this level is in corresponding $n$-dimensional Laplacian, we will arrive at the level $(n-k-1)$ with exactly the same form of the lower dimensional Hamiltonian. This completes the prove that the Hamiltonian, given as a generalized to the $n$-particle wave functions and using the matching conditions, it is not difficult to establish the differences between the $S$-matrix calculations for the NLS and LL models. Besides the technical difficulties associated with rapidly increasing, in each perturbation order, number of vertices in the first nontrivial order in the perturbation series. As we explained in Ref. 39, there are essential differences between the $S$-matrix calculations for the NLS and LL models. Besides the technical difficulties associated with rapidly increasing, in each perturbation order, number of vertices in the LL model, which leads to a complex diagrammatic analysis, there are conceptual difficulties related to the identification within the field-theoretic approach of the Bethe particles.

Here, however, we will establish the $S$-matrix factorization of the $su(1,1)$ LL model in a more direct fashion, following the results of the previous sections, where we have explicitly constructed the self-adjoint extensions and their continuity properties. Thus, having the exact expressions for the $n$-particle wave functions and using the matching conditions, it is not difficult to establish the $S$-matrix factorization. In fact, one can proceed in the same fashion as for the simpler NLS model (see, for example, Ref. 31). Indeed, the main result from Sec. III, that allows explicit calculation of the $n$-particle $S$-matrix, is that the matching conditions for the $n$-particle case (53) are exactly the ones for the two-particle case (26). In other words, it is enough to solve the equations for the two-particle case, in order to obtain the solution for the general $n$-particle case. This is, essentially, the $S$-matrix factorization.

IV. S-MATRIX FACTORIZATION

Although the $S$-matrix factorization is the underlying property of quantum integrable systems, it is quite difficult to analytically prove it using the standard perturbative calculations. Until now such calculations were fully performed in all orders only for the NLS model, where the $S$-matrix factorization was indeed proven—first for the three-particle scattering process, and then generalized to the $n$-particle scattering process. Recently, in Ref. 39 we have considered the $S$-matrix factorization following similar perturbative calculations, based on earlier two-particle $S$-matrix calculations of Ref. 40. We were able to show the three-particle $S$-matrix factorization in the first nontrivial order in the perturbation series. As we explained in Ref. 39, there are essential differences between the $S$-matrix calculations for the NLS and LL models. Besides the technical difficulties associated with rapidly increasing, in each perturbation order, number of vertices in the LL model, which leads to a complex diagrammatic analysis, there are conceptual difficulties related to the identification within the field-theoretic approach of the Bethe particles.
To construct explicit expressions, we follow the derivation of Ref. 31 for the NLS model. It is easy to see that for the LL model, the matching condition (28) leads to the following $n$-particle wave function:

$$f_n(x_1|p_1) = (\text{const}) \sum_{\{i\}} (-1)^{[i]} e^{\frac{i}{h} \sum_{i \neq j} p_{[i]} \cdot p_{[j]}} \prod_{i \neq j} \left( p_{[i]} - p_{[j]} - \frac{1}{2} i [p_{[j]} p_{[j]}] \right), \quad (59)$$

where $\{i\}$ denotes all possible permutations of $(1, \ldots, n)$. Thus, the $n$-particle wave function is factorized in terms of the two-particle wave functions. As a consequence, the $n$-particle scattering S-matrix has the form

$$S_n(p_1, \ldots, p_n) = \prod_{i \neq j} S_2(p_i, p_j), \quad (60)$$

where the two-particle scattering S-matrix for the $su(1,1)$ LL model has the form

$$S_2 = \frac{2(p_1 - p_2) - ip_1 p_2}{2(p_1 - p_2) + ip_1 p_2}. \quad (61)$$

Let us note that for the NLS model the expression for the wave function is similar to (59) form, where the third term in the brackets is a constant. Here, we have momenta product instead, which is the result of derivatives present in the interaction vertex (see, for details, Ref. 40). With this in mind, we can intuitively think of the LL model as the NLS model with momentum-dependent interaction.

V. CONCLUSION

We have considered the quantum integrable properties of the LL model and proposed a method to construct the quantum Hamiltonian. Most importantly, we achieve this directly in the continuous case by regularizing the ill-defined Hamiltonian and constructing the necessary self-adjoint extensions. This method allowed us to consistently derive the spectrum, which we show to coincide with the one following from the quantum inverse scattering method. We gave an explanation and derived in the most general $n$-particle case the puzzling construction of Sklyanin\(^24\) (for the particular two-particle sector) of the quantum-mechanical Hamiltonian action on the vectorlike state. The continuity properties of the functions involved in the construction of such states have also been carefully investigated. These properties are defined by the corresponding matching conditions and, as we have shown, lead to the S-matrix factorization property.

The particular difficulties of the LL model quantization are the ill-defined operator product of the local conserved charges, as well as the highly singular potential in the quantum-mechanical picture, which make it impossible the use of the trace identities in the quantum case. Thus, it is clear, that the method considered in this paper should be applicable to any continuous integrable model which has a singular nature. Since we have only considered the isotropic LL model, the next natural problem is to consider the anisotropic LL model, which is of great importance in the theory of integrable models. This, however, seems to be a more complex task, as the algebra of observables should be, for consistency, modified by hand in the inverse scattering method, forming the Sklyanin algebra. Let us remind that this algebra is naturally obtained from the lattice models, but its appearance in the continuous models is less clear. It would be interesting to give a direct derivation of the Sklyanin algebra without appealing to the lattice version, since in more complex continuous integrable models the construction of the corresponding lattice models, as we have discussed in Sec. I, is generally a quite complex task that has not been well understood even for simple models.

Another important problem, also discussed in Sec. I, is to investigate the $su(2)$ LL model. Let us remind that in the Sklyanin’s original paper\(^24\) as well as in our work, only the quantization of the $su(1,1)$ LL model is considered. This is done to have physically meaningful states in the chosen representation for the states, consistent with the ferromagnetic choice of the vacuum.
Constructing representations in the $su(2)$ case with positively defined metric seems to be a more complex task which has not so far been considered in connection with the quantization of the LL model. It is known that such representations are possible to construct, but the vacuum will not be ferromagnetic anymore. Thus, the algebraic Bethe ansatz is not applicable in this case, and one has to consider more sophisticated methods of finding algebraic solutions, much like for the sinh-Gordon model.36,37

Although we have considered only regularization and diagonalization of the first nontrivial conserved charge (the Hamiltonian), integrability implies the conservation of the infinite tower of charges that should be possible to regularize in the manner similar to the method proposed in this paper. We do not currently know whether it is possible to do in a unified manner or each charge should be considered separately. Clearly, the questions posed above for the LL mode, as the simplest representative of associated difficulties, will appear in other more interesting continuous integrable model. As an example, we mention the recently discovered Alday–Arutyunov–Frolov fermionic model, which appears in the $su(1,1)$ subsector of the AdS$_3 \times S^5$ strings.44 There we expect the similar difficulties to appear in the quantization process, as the singular nature of the fermionic interaction terms will clearly require construction of the self-adjoint extensions and careful consideration of the conserved charges. These and other related problems are currently under investigation.

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