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PARTIAL STABILITY FOR A CLASS OF NONLINEAR SYSTEMS

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Abstract. This paper studies a nonlinear, discrete-time matrix system arising in the stability analysis of Kalman filters. These systems present an internal coupling between the state components that gives rise to complex dynamic behavior. The problem of partial stability, which requires that a specific component of the state of the system converge exponentially, is studied and solved. The convergent state component is strongly linked with the behavior of Kalman filters, since it can be used to provide bounds for the error covariance matrix under uncertainties in the noise measurements. We exploit the special features of the system—mainly the connections with linear systems—to obtain an algebraic test for partial stability. Finally, motivated by applications in which polynomial divergence of the estimates is acceptable, we study and solve a partial semistability problem.

Key words. stability, nonlinear systems, matrix analysis, Kalman filter stability

AMS subject classifications. 93D99, 70K20, 93B99, 15A99, 93E11

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1. Introduction. Partial stability (PS) refers to the class of problems that deal with stability of some state components with respect to (w.r.t.) those same components or w.r.t. all state components, or even w.r.t. some (nonfixed) components. It has been studied for linear and nonlinear systems from different perspectives [2, 6, 9, 10, 14], and vector Lyapunov functions constitute one of the first and most common tools in the literature, which can be traced back to the fifties [12]. PS arises naturally in many applications, e.g., in situations where some of the variables are not important as far as the operation and performance are concerned, or are not essential at all (in a sense related to model order reduction problems); see [14] for a quite complete assessment of specific problems in PS literature.

In this paper we deal with a discrete-time matrix system arising in the analysis of the error covariance matrix $V_k$ of Kalman filters under noise measurement uncertainties; see [3]. Therein it is shown that, assuming the filter gains are calculated taking into account an initial error covariance $\Sigma$ that differs from the actual one $V_0$, if the “incorrect” calculated error covariances are bounded, then for each $0 \leq \zeta < 1$ there exist $\tau > 0$ and $M = M' \geq 0$, providing the bounds

$$\left(1 - \tau\right)Z_k - \zeta^{-k}M \leq V_k \leq \left(1 + \tau\right)Z_k + \zeta^{-k}M, \quad k \geq 0,$$

for the actual error $V_k$, where $Z_k$ is a component of the state $(Z_k, X_k)$ of the system

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described by the equations

\[
\begin{align*}
\Theta : & \quad \left\{ \begin{array}{l}
Z_{k+1} = H_k A Z_k A^T H_k', \\
X_{k+1} = A X_k A', \\
(Z_0, X_0) = (H_0 V_0 H_0', \Sigma),
\end{array} \right.
\end{align*}
\]

where \(Z_k, X_k, V_0,\) and \(\Sigma\) are symmetric positive semidefinite matrices and the square matrix \(A\) is assumed to be known. \(H_k, \ k \geq 0,\) stands for the orthogonal projection onto the null space of \(X_k.\) For instance, when \(\ker\{\Sigma\} \subset \ker\{V_0\}\) (e.g., when \(\Sigma = V_0\) or \(\Sigma > 0\)), we have \(H_0 V_0 H_0' = 0,\) yielding \(Z_k = 0, \ k \geq 0.\) We refer to \(H_k\) as the coupling projections, since it couples the dynamics of the \(Z\)-component with the trajectory of the \(X\)-component.

From the standpoint of the Kalman filter application, it is important to characterize, in terms of \(A\) and \(\Sigma,\) the existence, for each \(V_0,\) of \(0 \leq \beta < 1\) and \(Z\) such that \(Z_k \leq \beta^k Z, \ k \geq 0,\) or, for each \(V_0\) and \(0 \leq \zeta < 1,\) of \(Z\) such that \(\zeta^k Z_k \leq Z,\) meaning that the \(Z\)-component cannot diverge exponentially. Moreover, since \(X_0 = \Sigma\) is fixed, we are interested only in the behavior of the \(Z\)-component w.r.t. \(V_0.\) We refer to these problems as PS and partial semistability\(^1\) (PSS), respectively. PSS is relevant in situations where polynomial divergence of the Kalman estimates is acceptable.

Approaches for PS and PSS of general nonlinear systems can be employed, in principle. However, they are too general to yield an easy-to-test algebraic condition. The available results for PS of linear systems do not apply directly to \(\Theta,\) and it is worth mentioning that it is inappropriate to deal with the problem assuming that \(H_k\) are general projections, not connected with \(X,\) in order to retrieve linearity; in fact, in such a modified setting, \(Z_k\) can diverge exponentially, whereas \(A\) is stable\(^2\) (\(A\) stable implies PS; see Remark 2). There is no available result specialized to system \(\Theta.\)

This paper exploits the special features of \(\Theta,\) for instance, the \(X\)-component obeys a linear difference equation and the coupling is via orthogonal projections; see other interesting properties in Proposition 1. We make use of a sequence of transformations \(W_k, \ k \geq 0,\) playing the role of time-variant bases that allow one to characterize the convergence of the null space of \(X_k,\) as in Lemmas 6 and 7, and allow for adequate evaluations for the coupling projections in Lemma 9. Based on these evaluations, we show that \(\Theta\) is PSS if and only if the structural relation

\[
\ker\{J_\Sigma J^{-1}\} \cap \mathcal{J} = \{0\}
\]

holds, where \(J\) is a similarity transformation such that \(J A J^{-1}\) is in Jordan form and \(\mathcal{J}\) stands for the unstable subspace\(^3\) of \(J A J^{-1}.\) Recalling from linear systems theory that \((A, \Sigma)\) semistabilizable can be interpreted as requiring that \(\Sigma\) excites the unstable space of \(A,\) the interpretation of (2) is that \(\Sigma\) has to “completely excite” the unstable space of \(A.\) Regarding PS, a similar condition holds, where \(\mathcal{J}\) is replaced with \(\mathcal{J}^+_S\) and \(\mathcal{J}^+_S\) is the stable space of \(A.\) These conditions are compared with classical notions of stabilizability and semistabilizability of \((A, \Sigma);\) see Remark 2. Moreover, the conditions can be employed for “stabilization,” for instance, to obtain a \(\Sigma\) that provides PS or PSS; see Remark 3.

Apart from inherent theoretical significance, the derived conditions pave the way for obtaining sharp conditions for stability and semistability of Kalman filters [3, 4],

\(^1\)Following the terminology of [1].
\(^2\)For example, set \(A = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\) and \(H_k = V_0 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ k \geq 0.\) \(A\) is stable and \(Z_k\) diverge.
\(^3\)Please see section 2 for definitions.
which is a highly important issue, since existing conditions are conservative as they are
either necessary or sufficient, or rely on additional assumptions, such as the existence
of limiting stationary filters; see [7, 11, 13, 15].

The paper is organized as follows. Section 2 presents definitions and preliminary
results. Section 3 introduces the sequence of transformations that allow us to derive
a simple structure for A and to simplify the evaluation of the projections H_k. These
results allow us to obtain testable conditions for PS and PSS is section 4. Finally,
section 5 provides some conclusions.

2. Definitions and preliminary results. Let \( \mathbb{R}^n \) denote the nth dimensional
Euclidean space. Let \( \mathbb{D} \) (respectively, \( \overline{\mathbb{D}} \)) be the open (closed) unit disk. Let \( e_i \),
\( i = 1, \ldots, n \), be the canonical basis of \( \mathbb{R}^n \). \([v_1, \ldots, v_m] \) stands for the vector space
spanned by \( v_1, \ldots, v_m \in \mathbb{R}^n \). For vector subspaces \( \mathcal{E} \) and \( \mathcal{F} \), \( \mathcal{E} \perp \mathcal{F} \) means that \( \mathcal{E} \)
and \( \mathcal{F} \) are orthogonal, \( \mathcal{E}^\perp \) is such that \( \mathcal{E}^\perp \perp \mathcal{E} \); \( \mathcal{E} \oplus \mathcal{F} \) is the direct sum of \( \mathcal{E} \)
and \( \mathcal{F} \), and \( \mathcal{E} \ominus \mathcal{F} = \mathcal{E} \cap \mathcal{F}^\perp \). Let \( \mathcal{R}^r \) (respectively, \( \mathcal{R}' \)) represent the normed linear
space formed by all \( r \times s \) real matrices (respectively, \( r \times r \) ) and \( \mathcal{R}^r(\mathcal{R}^0) \) the cone
\( \{ U \in \mathcal{R}' : U = U' \} \) (the closed convex cone \( \{ U \in \mathcal{R}' : U = U' \geq 0 \} \)), where \( U' \)
denotes the transpose of \( U \). For \( U \in \mathcal{R}^n \), \( \lambda_i(U) \), \( i = 1, \ldots, n \), stands for an eigenvalue
of \( U \). \( \lambda_i(U) \) is referred to as a semistable (respectively, stable) eigenvalue when it
lies in \( \mathbb{D}(\overline{\mathbb{D}}) \). The associated eigenvector \( v \in \mathbb{R}^n \) is semistable (stable); otherwise it
is unstable. The space spanned by all stable eigenvectors is referred to as the stable
subspace of \( U \), and similarly for semistable and unstable semispaces.

Regarding the system \( \Theta \) and its state trajectory \((Z_k, X_k)\), \( k \geq 0 \), we employ
the notation \( Z_k(V_0) \) and \( X_k(\Sigma) \) to emphasize the dependence on \( V_0 \) and \( \Sigma \); when the
dependence of \( Z_k \) on \( \Sigma \) (indirectly via the coupling projections) is relevant, we employ
the notation \( Z_k(V_0, \Sigma) \). The coupling projections \( H_k \) give rise to the nonlinearities of
\( \Theta \); for example, with \( A = I \) and \( V_0 \neq 0 \) we have \( Z_k(V_0, I) + Z_k(V_0, 0) = V_0 \neq 0 = Z_k(2V_0, I) \), \( k \geq 0 \). Some useful features of system \( \Theta \) are presented in what follows.

**PROPOSITION 1.** Consider system \( \Theta \) and, for \( U \in \mathcal{R}^n \), let \( U^* \) stand for the
pseudoinverse of \( U \). The following statements hold:

1. \( H_k = I - X_k^* X_k = I - (\zeta X_k)^* (\zeta X_k) \), \( k \geq 0 \), \( \zeta \neq 0 \).
2. \( (Z_k(U), X_k(\Sigma)) = (Z_k(U), \zeta X_k(\Sigma)) \), \( k \geq 0 \), \( \zeta \geq 0 \).
3. \( U \geq U_0 \), then \( Z_k(U_1) \geq Z_k(U_0) \), \( k \geq 0 \).
4. \( \Sigma \geq \Sigma_0 \), then \( X_k(\Sigma_1) \geq X_k(\Sigma_0) \) and \( Z_k(V_0, \Sigma_1) \leq Z_k(V_0, \Sigma_0) \), \( k \geq 0 \).

The stability notions considered in this paper are as follows.

**DEFINITION 1.** Consider system \( \Theta \). We say that \((A, \Sigma) \) is partially semistable
(PSS) if, for each \( 0 \leq \zeta < 1 \) and \( V \in \mathcal{R}^n \), there exists \( Z \in \mathcal{R}^n \) such that \( \zeta^k Z_k(V) \leq Z \), \( k \geq 0 \). We say that \((A, \Sigma) \) is partially stable (PS) if, for each \( V \in \mathcal{R}^n \), there
exists \( 0 \leq \beta < 1 \) and \( Z \in \mathcal{R}^n \) such that \( Z_k \leq \beta^k Z \), \( k \geq 0 \).

**Example 1.** Consider the system \( \Theta \) with

\[
A = \begin{bmatrix} d & 1 \\ 0 & d \end{bmatrix}, \quad \Sigma = \sigma \sigma'.
\]

Set \( d = -1 \) and \( \sigma = [0 \ 0]' \). From (1) we have that \( X_k = 0 \), \( k \geq 0 \). Direct inspection
of PSS and PS via Definition 1 is virtually impossible, as it involves exhaustive searches
for \( Z \), for each \( V \) and \( \zeta \). Moreover, there is no evidence on how to modify the
parameters (e.g., \( \sigma \)) in order to achieve PSS or PS. In Example 5 we shall see that
\((A, \Sigma) \) is PSS (and not PS), despite the fact that \( Z_k \) can diverge with polynomial rate;
for instance, with \( V = vv' \) and \( v = e_2 \), (1) yields
\[
Z_k(V) = A^k V A^k = \begin{bmatrix} k^2 & -k \\ -k & 1 \end{bmatrix}.
\]

**Lemma 1.** Consider system \( \Theta \). The following statements hold:

(i) \( (A, \Sigma) \) is PSS if and only if, for each \( \rho > 1 \), there exists \( \tilde{Z} \in \mathbb{R}^{n_0} \) such that \( Z_k(I) \leq \rho^k \tilde{Z}, \) \( k \geq 0 \).

(ii) \( (A, \Sigma) \) is PS if and only if there exist \( \tilde{Z} \in \mathbb{R}^{n_0} \) and \( 0 \leq \gamma < 1 \) such that \( Z_k(I) \leq \gamma^k \tilde{Z}, \) \( k \geq 0 \).

*Proof of (i).* (Necessity.) It follows by setting \( V = I \) in Definition 1.

(Sufficiency.) For each \( V \in \mathbb{R}^{n_0} \) we can pick \( \kappa > 0 \) such that \( \kappa V \leq I \) and Proposition 1 (iii) yields \( Z_k(\kappa V) \leq Z_k(I) \leq \rho^k \tilde{Z} \), which leads to \( \rho^{-k}Z_k(V) \leq \kappa^{-1} \tilde{Z} \).

*Proof of (ii).* It is similar to the proof of (i) and is not presented.

Consider now the linear time-varying system related to the dynamics of the \( Z \)-component of \( \Theta \), defined by
\[
\Theta_Z : \begin{cases} 
  z_{k+1} = H_k Az_k, & k \geq 0, \\
  z_0 = H_0 z, 
\end{cases}
\]

where \( z_k \in \mathbb{R}^n \) is the state and \( z \in \mathbb{R}^n \). Not surprisingly, PSS and PS of \( (A, \Sigma) \) are strongly connected to semistability and stability of \( \Theta_Z \), as stated in the following lemma.

**Lemma 2.** Consider systems \( \Theta \) and \( \Theta_Z \). The following statements hold:

(i) \( (A, \Sigma) \) is PSS if and only if, for each \( z \in \mathbb{R}^n \) and \( 0 \leq \zeta < 1 \), there exist \( \alpha \geq 0 \) and \( 0 \leq \beta < 1 \) such that \( \|\zeta^k z_k\| \leq \alpha \beta^k \).

(ii) \( (A, \Sigma) \) is PS if and only if for each \( z \in \mathbb{R}^n \) there exist \( \alpha \geq 0 \) and \( 0 \leq \beta < 1 \) such that \( \|z_k\| \leq \alpha \beta^k \).

*Proof of (i).* We employ the notation \( z_k(z), \alpha_z, \) and \( \beta_z \) to emphasize the dependence on \( z \).

(Necessity.) Let \( \iota > 1 \) and, for each \( 0 \leq \zeta < 1 \), let \( \rho = \zeta^{-2\iota} \). From Lemma 1, there exists \( \tilde{Z} \) such that \( Z_k(I) \leq (\zeta^{-2\iota})^k \tilde{Z} \). Equivalently,
\[
\zeta^{2k} Z_k(I) \leq \iota^k \tilde{Z}.
\]

Consider \( z \) such that \( \|z\| \leq 1 \). Note that \( Z_0(I) = H_0 H_0' \geq H_0 z z' H_0' = z_0 z_0' \) and that employing (1) and (4) recursively yields \( Z_k \geq z_k z_k' \), \( k \geq 0 \). Then (5) leads to \( \zeta^{2k} z_k z_k' \leq \iota^k \tilde{Z} \), and taking the trace we obtain
\[
\|\zeta^k z_k\|^2 \leq \iota^k \text{tr}(\tilde{Z}), \quad \|z\| \leq 1.
\]

Now consider \( \|z\| > 1 \). Note from (4) that \( z_k(\|z\|^{-1} z) = \|z\|^{-1} z_k(z) \), which allows us to employ (6) to evaluate
\[
\|\zeta^k z_k\|^2 = \|\zeta^k z_k(\|z\|^{-1} z)(\|z\|^{-1} z)\|^2 \leq \iota^k \text{tr}(\tilde{Z}) \|z\|^2.
\]

From (6) and (7) we have that, for each \( z \in \mathbb{R}^n \), \( \|\zeta^k z_k\|^2 \leq \iota^k \text{tr}(\tilde{Z}) \max(1, \|z\|^2) \).

(Sufficiency.) For each \( \gamma > 1 \), let \( \zeta = \sqrt{\gamma^{-1}} \) and note that for each \( z \), by hypothesis, \( z_k(z) z_k(z)' \leq \zeta^{-2k} \beta_z^2 \alpha_z^2 I = \gamma^k \beta_z^2 \alpha_z^2 I \). Then, for each \( z = e_i, i = 1, \ldots, n \), we can write
\[
z_k(e_i) z_k'(e_i) \leq \gamma^k \beta_{e_i}^2 \alpha_{e_i}^2 I.
\]
Since $H_k$ is an orthogonal projection, employing (1) we obtain $Z_0(I) = H_0H_0' \leq I = z_0(e_1)z_0'(e_1) + \cdots + z_0(e_n)z_0'(e_n)$, and it is simple to check by induction that

$$Z_k(I) \leq z_k(e_1)z_k'(e_1) + \cdots + z_k(e_n)z_k'(e_n).$$

Equations (8) and (9) lead to

$$Z_k(I) \leq \gamma^k \beta_1^2 \alpha_1^2 I + \cdots + \gamma^k \beta_n^2 \alpha_n^2 I \leq \gamma^k \bar{\beta}^2 n \bar{\alpha}^2 I,$$

where $\bar{\alpha} = \max(\alpha_{e_1}, \ldots, \alpha_{e_n})$ and $\bar{\beta} = \max(\beta_{e_1}, \ldots, \beta_{e_n})$. Since $\beta_{e_i} < 1$ and $\bar{\beta} < 1$, (10) leads to $Z_k(I) \leq \gamma^k (n \bar{\alpha}^2 I)$ and Lemma 1 completes the proof.

Proof of (ii). It is similar to the proof of (i), replacing $\bar{\varepsilon} > 1$, $0 \leq \zeta < 1$, and $\gamma > 1$ with $0 \leq \bar{\varepsilon} < 1$, $\zeta = 1$, and $\gamma = 1$, respectively. \qed

Similarly to the sequence $x_k$ connected with the $Z$-component of $\Theta$, we introduce a vector sequence related to $X$, as follows. Consider the solution $X_k = A^k \Sigma A^{k'}$ for the $X$-component. Introduce the rank-one decomposition

$$\Sigma = \sigma_1 \sigma_1' + \cdots + \sigma_{r_{\Sigma}} \sigma_{r_{\Sigma}}',$$

where $r_{\Sigma}$ stands for the rank of $\Sigma$, and the linear system defined by

$$\Theta_X : x_k(\sigma) = A^k \sigma.$$

It is simple to check that

$$X_k = x_k(\sigma_1)x_k(\sigma_1)' + \cdots + x_k(\sigma_{r_{\Sigma}})x_k(\sigma_{r_{\Sigma}})'$$

and $H_k$ is the orthogonal projection onto $[x_k(\sigma_1), \ldots, x_k(\sigma_{r_{\Sigma}})]^\perp$.

3. Evaluations for the coupling projections. The spaces spanned by the trajectory $x_k = A^k \sigma$ play an important role in this paper, because they drive the projection $H_k$. We now present certain characterizations for convergence of these spaces. Note that, taking into account the original basis, there may be no convergence for $[x_k]$; see Examples 2 and 4. In this paper we employ the bases introduced as follows, related to Jordan forms [8], in view of the fact that they lead to a simpler characterization for $[x_k]$, and despite the drawback of an inherent time dependence.

**Proposition 2.** For each $A \in \mathbb{R}^n$ there is a sequence of transformations $W_k$, $k \geq 0$, such that $A = W_{k+1}^{-1} \bar{A} W_k$ and $A^k = W_k^{-1} \bar{A}^k W_0$, $k \geq 0$, with

$$\bar{A} = \text{diag}(A(\eta_1), \ldots, A(\eta_j)),$$

where $A(\eta_i), 0 \leq i \leq j \leq n$, is an upper triangular Jordan block with eigenvalue $\eta_i$, and $\eta_i$ is a real nonnegative number, corresponding to certain eigenvalues $\lambda_\ell(A)$, $0 \leq \ell \leq n$, with $|\lambda_\ell(A)| = \eta_i$, ordered in such a manner that $\eta_i \geq \eta_j$ whenever $i \geq j$. Moreover, there exists $\kappa$, $0 \leq \kappa < 1$, such that $(1 - \kappa) \leq \|W_k\| \leq (1 + \kappa)$, $k \geq 0$.

The bases of Proposition 2 are employed throughout the paper, hence we introduce the following notation. Unless otherwise stated, for any $V \in \mathbb{R}^{n,r}$ and $v \in \mathbb{R}^n$, we define $\bar{V} \in \mathbb{R}^{n,r}$ and $\bar{v} \in \mathbb{R}^n$ as $V = W_0 \bar{V}$ and $v = W_0 \bar{v}$. For instance, we denote $W_0 \sigma$ simply by $\bar{\sigma}$. The matrix $A$ associated with the transformation $W_0$ is usually clear from the context; otherwise we employ the explicit notation $W_0(A)$. For $\sigma, z \in \mathbb{R}^n$, define $z_k, \bar{x}_k \in \mathbb{R}^n$, $k \geq 0$, by

$$z_{k+1} = (\bar{H}_k \bar{A})z_k, \quad k \geq 1, \quad \bar{z}_0 = \bar{H}_0 \bar{z}, \quad \bar{x}_k(\sigma) = \bar{A}^k \bar{\sigma}, \quad k \geq 0,$$
Lemma 4; see, e.g., [1]. Another useful connection is as follows. Let \( J \) be the similarity matrix for which \( JAJ^{-1} \) is the Jordan form of \( A \) and let \( J \) stand for the vector subspace spanned by the unstable eigenvectors of \( JAJ^{-1} \). Then, for each \( \sigma \in \mathbb{R}^n \), the

\[
\tilde{H}_k = W_k H_k W_k^{-1}.
\]

**Example 2.** Consider

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \tilde{A} = I, \quad W_{2\ell} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad W_{2\ell+1} = I, \quad \ell \geq 0.
\]

It is simple to check that the statements of Proposition 2 are satisfied. For \( \sigma = [1 \ 1]' \), \( x_k = A^k \sigma \) is in the form \( x_{2\ell} = [1 \ 1]' \) and \( x_{2\ell+1} = [1 \ -1]' \), \( \ell \geq 0 \); hence the spaces spanned by \([x_k]\) do not converge in any sense as \( k \to \infty \). On the other hand, \( \bar{x}_k = \tilde{A}^k \bar{\sigma} = W_0 \sigma = [1 \ -1]' \), \( k \geq 0 \). The matrix \( A \) is in Jordan form, making clear that the Jordan form is not convenient for the characterization of convergence of \([x_k]\) that we seek. Note that \( \tilde{A} \) is also in Jordan form, but is not similar to \( A \).

Convergence of state trajectories is preserved, as stated in the next result.

**Lemma 3.** The following statements hold:

(i) \( z_k = W_{k-1}^{-1} \bar{z}_k, x_k(\sigma) = W_k \bar{x}_k(\sigma), k \geq 0 \).

(ii) There exists \( \kappa, 0 \leq \kappa < 1 \), such that

\[
(1 + \kappa)^{-1} \| \bar{z}_k \| \leq \| z_k \| \leq (1 - \kappa)^{-1} \| \bar{z}_k \|
\]

and \( (1 - \kappa) \| \bar{x}_k(\sigma) \| \leq \| x_k(\sigma) \| \leq (1 + \kappa) \| \bar{x}_k(\sigma) \|, k \geq 0 \).

**Proof of (i).** We have from Proposition 2 that \( A = W_{k-1}^{-1} \tilde{A} W_{k-1}, k \geq 1 \); substituting this equality and (13) in (4) yields

\[
z_k = (H_{k-1} A) \cdots (H_1 A)(H_0 v) = (W_{k-1}^{-1} \tilde{H}_{k-1} W_{k-1} W_{k-1}^{-1} \tilde{A} W_{k-2}) \cdots (W_0^{-1} \tilde{H}_0 W_0 v)
\]

\[
= W_{k-1}^{-1} (\tilde{H}_{k-1} \tilde{A}) \cdots (\tilde{H}_1 \tilde{A}) \tilde{H}_0 W_0 v = W_{k-1}^{-1} \bar{z}_k.
\]

The proof for the second statement of (i) is analogous.

**Proof of (ii).** Accordingly to Proposition 2, there is a \( 0 \leq \kappa < 1 \) such that

\[
\| W_k \| \leq (1 + \kappa), k \geq 0,
\]

and assertion (i) allows us to write \( \| z_k \| \leq \| W_{k-1}^{-1} \| \| \bar{z}_k \| \leq (1 - \kappa)^{-1} \| \bar{z}_k \| \). A similar evaluation yields \( \| z_k \| \geq (1 + \kappa)^{-1} \| \bar{z}_k \| \). The proof for the second statement of (ii) is analogous.

**Proposition 3.** Consider system \( \Theta_Z \) and the system \( \Theta_{\zeta Z} \) that arises by replacing the matrix \( A \) with \( (\zeta A) \), \( \zeta > 0 \), and let \( z_{\zeta, k} \) be the corresponding trajectory. Then \( \zeta^k z_{\zeta, k} = z_{\zeta, k} \) and, similarly, \( \zeta^k \bar{z}_{\zeta, k} = \bar{z}_{\zeta, k} \).

**Proof.** The first statement follows from Proposition 1 (i)–(iii). Moreover, the first statement and Lemma 3 (i) yield \( \zeta^k W_{k-1}^{-1}(A) \bar{z}_k = W_{k-1}^{-1} (\zeta A) \bar{z}_{\zeta, k} \); hence the second statement follows from the fact that one can always set \( \bar{W}_k (\zeta A) = W_k (A) \) (even for \( \zeta = 0 \)).

Consider now \( \tilde{A} \) and let \( e_1, \ldots, e_q \) and \( e_{q+1}, \ldots, e_d \) be the associated unstable eigenvectors and semistable eigenvectors, respectively. Introduce the subspaces

\[
U = [e_1, \ldots, e_q], \quad S = [e_1, \ldots, e_d].
\]

Of course, \( U^\perp = [e_{q+1}, \ldots, e_d] \) and \( S^\perp = [e_{q+1}, \ldots, e_d] \). The block structure of \( \tilde{A} \) in Proposition 2 allows for the next invariance results.

**Lemma 4.** \( U, U^\perp, S, \) and \( S^\perp \) are \( \tilde{A} \)-invariant.

Note that \( \tilde{A} \) is in Jordan form [8], leading to several links with available results for Jordan forms. For example, there are invariance results similar to the ones of Lemma 4; see, e.g., [1]. Another useful connection is as follows. Let \( J \) be the similarity matrix for which \( JAJ^{-1} \) is the Jordan form of \( A \) and let \( J \) stand for the vector subspace spanned by the unstable eigenvectors of \( JAJ^{-1} \). Then, for each \( \sigma \in \mathbb{R}^n \), the
projection of $J\sigma$ onto $J$ is zero if and only if the projection of $\bar{\sigma} = W_0\sigma$ onto $U$ is zero, yielding the following result, given without proof, which is useful for representing the main results in terms of Jordan forms.

**Lemma 5.** $\ker\{W_0\Sigma W_0'\} \cap U = \{0\}$ if and only if $\ker\{J\Sigma J'\} \cap J = \{0\}$.

The spaces spanned by $x_k$ may not converge in any sense (see Example 2), which implies that there may be no convergence for the projections $H_k$. However, the convenient structure of $\bar{A}$ provides that $|\bar{x}_k|$ always converge in a certain sense, allowing us to derive approximation results for $\bar{H}$. In order to make the convergence notion precise, we define, for the (nontrivial) vector subspaces $U$ and $V$, the quantity

$$\theta_{V}(U) = \max_{v \in V, v \neq 0} \min_{u \in U, u \neq 0} 1 - \frac{1}{\|u\|\|v\|} u'v.$$  

Note from the structure of $\bar{A}$ that if $\sigma \in \mathbb{R}^n$ is such that $\eta$ is the largest eigenvalue for which $\sigma'v \neq 0$, where $v$ is an eigenvector associated with $\eta$, and assuming $\eta$ unique (i.e., no other eigenvalue of $\bar{A}$ equals $\eta$), then there are $\pi \geq 0$ and $0 \leq \rho < 1$ such that $\theta_{\bar{S}_n}(\bar{x}_k(\sigma)) \leq \pi \rho^k$, where $\bar{S}_n$ is the space spanned by the eigenvectors associated with $\eta$. This signifies that $\bar{x}_k(\sigma)$ and $\bar{S}_n$ “align” with exponential rate. Moreover, there is a $\varphi > 0$ such that $\theta_{\mathbb{R}^n \cap \bar{S}_n}(\bar{x}_k(\sigma)) \geq \varphi$ for a sufficiently large $k$. One can explore the convenient block structure of $\bar{A}$ to obtain the more general characterization given in Lemmas 6 and 7 without proof; recall that $\bar{\sigma} = W_0\sigma$ and $\bar{x}_k(\sigma) = A^k\bar{\sigma} = W_kA^k\sigma$.

**Lemma 6.** Consider $\sigma_j \in \mathbb{R}^n, j = 1, \ldots, m$, and assume $S$ is nontrivial. If $\ker\{\bar{\sigma}_1\bar{\sigma}_1' + \cdots + \bar{\sigma}_m\bar{\sigma}_m'\} \cap S = \{0\}$, then there exist $\pi \geq 0$ and $0 \leq \rho < 1$ such that

$$\theta_{S}(\bar{x}_k(\sigma_1), \ldots, \bar{x}_k(\sigma_m)) \leq \pi \rho^k.$$  

Conversely to Lemma 6, if $\sigma_j$ does not “completely excite” the subspace $S$, then the space spanned by $\bar{x}_k(\sigma_1)$ does not “align” with $S$. It is convenient for later reference to formalize this in terms of $U$ rather than $S$.

**Lemma 7.** Consider $\sigma_j \in \mathbb{R}^n, \sigma_j \neq 0, j = 1, \ldots, m$. If $\ker\{\bar{\sigma}_1\bar{\sigma}_1' + \cdots + \bar{\sigma}_m\bar{\sigma}_m'\} \cap U \neq \{0\}$, then there exist $\varphi > 0$ and a subspace $F$ spanned by eigenvectors of $A$ such that $U \cap F$ is nontrivial and

$$\theta_{F}(\bar{x}_k(\sigma_1), \ldots, \bar{x}_k(\sigma_m)) \leq o(k), \quad k \geq 0,$$

$$\theta_{U \cap F}(\bar{x}_k(\sigma_1), \ldots, \bar{x}_k(\sigma_m)) \geq \varphi, \quad k \geq 1,$$

where $o(\cdot)$ is a nonnegative-valued, strictly decreasing function.

**Example 3.** Consider the system $\Theta$ of Example 1 with $\sigma = e_2$. The statements of Proposition 2 hold with $\bar{A} = A$ and $W_k = I, k \geq 0$, whenever $d \geq 0$, or

$$\bar{A} = -A, \quad W_{2\ell} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad W_{2\ell+1} = -W_{2\ell}, \quad \ell \geq 0, \quad d < 0.$$  

Consider $d$ such that $|d| > 1$. Clearly, $\bar{\sigma} = \sigma$ and $\mathcal{U} = \mathbb{R}^n$ satisfy the hypothesis of Lemma 7. As $k \to \infty$, $\bar{x}_k(\sigma)$ “aligns” with $F = \{e_1\}$. Figure 1 illustrates the behavior of $\theta$; note from the detail presented in Figure 1 (ii), for $d = 10$, that the convergence is slower in the interval $100 \leq t \leq 200$ than in the interval $0 \leq t \leq 100$, suggesting it is not exponential. The case when $|d| \leq 1$ is not addressed by Lemmas 6 or 7 (note that $\mathcal{U}$ is trivial; for $|d| < 1, \mathcal{S}$ is trivial and, for $|d| = 1, \sigma$ does not completely excite $S = \mathbb{R}^n$).

The coupling projections $\bar{H}$ differ from $H$ as they are are not orthogonal, because of the “distortion” introduced by the new bases (see Example 4), but they are similar
to $H$ in the sense that $\bar{H}v = 0$ whenever $Hv = 0$. We shall need the following related result.

Lemma 8. Consider the rank-one decomposition (11) for $\Sigma$. The projections $\bar{H}_k$, $k \geq 0$, are such that $H_kv = 0$ for $v \in [\bar{x}_k(\sigma_1), \ldots, \bar{x}_k(\sigma_{r_2})]$.

Proof. Note that $\bar{H}_k \bar{x}_k(\sigma_j) = W_k \bar{H}_k W_k^{-1} W_k x_k(\sigma_j)$ for certain scalars $\pi_j$, $j = 0, \ldots, r_2$, and, for each term of this sum, one can employ Lemma 3 (i) to evaluate $\bar{H}_k \bar{x}_k(\sigma_j) = W_k \bar{H}_k W_k^{-1} W_k x_k(\sigma_j) = W_k H_k x_k(\sigma_j) = 0$, $j = 0, \ldots, r_2$, since $H_k x_k(\sigma_j) = 0$ by definition of $\bar{H}_k$.

As $\bar{x}_k(\sigma_j)$, $0 \leq j \leq r_2$, aligns with $S$ ($F$, respectively) as stated in Lemma 7 (Lemma 6, respectively), we have that the projections $\bar{H}_k$ onto $[\bar{x}(\sigma_1), \ldots, \bar{x}(\sigma_{r_2})]$ “tend to align” with the orthogonal projection onto $S^\perp$ ($F^\perp$, respectively), which allows us to obtain the approximation results that will be useful for section 4. We present these results in the next lemma, in which $S$, $T$, and $U$ denote the orthogonal projections onto $S^\perp$, $F^\perp$, and $U \cap F$, respectively.

Lemma 9. If $\ker\{W_0 \Sigma W_0^* \cap S = \{0\} \}$ and $\ker\{W_0 \Sigma W_0^* \cap S^\perp \} = S^\perp$, then there exist $\pi \geq 0$, $0 \leq \rho < 1$, such that, for $k \geq 0$,

(i) $\|S(I - \bar{H}_k)\| \leq \pi \rho^k \|v\|$ and $\|\bar{H}_k(I - S)\| \leq \pi \rho^k \|v\|$.

If $\ker\{W_0 \Sigma W_0^* \cap U \neq \{0\}\}$, then there exist $\delta, \lambda > 0$ and a nonnegative-valued, strictly decreasing function $o(\cdot)$ such that, for $k \geq 0$,

(ii) $\|T(I - \bar{H}_k)\| \leq o(k) \|v\|$ and $\|\bar{H}_k(I - T)\| \leq o(k) \|v\|$;

(iii) $\|((U^A)^{k+1} Uv\| \geq (1 + \delta) \|Uv\|$;

(iv) $T A U v = U A U v$;

(v) $\|\bar{H}_k U v\| \leq \lambda \|U v\|$.

Proof. (i). Lemma 6 leads to the result, provided $S$ is nontrivial: for trivial $S$ it is simple to check that $S = H_k = I$ and the result holds with $\pi = 0$. (ii) Lemma 8 can be employed when $\Sigma \neq 0$. The case with trivial $\Sigma$ leads to $T = H_k = I$ and $o(k) = 0$, $k \geq 0$. (iii) It follows from the fact that $U$ is the projection onto $U \cap \mathcal{F}$, which is spanned by unstable eigenvectors of $A$; moreover, $(1 + \delta)$ equals the minimal of these eigenvalues. (iv) $U \cap \mathcal{F}$ is not necessarily $A$-invariant in general, but one can easily check from the structure of $A$ that, for $w \in U \cap \mathcal{F}$, $Aw \in U$, in such a manner that the component of $Aw$ in $U^\perp$ is zero and $T A w = U A w$. (v) It follows from the facts that $\bar{H}_k = W_k H_k W_k^{-1}$ and $(1 - \kappa) \leq \|W_k\| \leq (1 + \kappa)$ for some $0 \leq \kappa < 1$, as in Lemma 2.

Remark 1 (condition $\ker\{W_0 \Sigma_1 W_0^* \cap S^\perp \} = S^\perp$ in Lemma 9). In order to show that $\ker\{W_0 \Sigma_1 W_0^* \cap S = \{0\}\}$ is a sufficient condition for PSS, we may initially consider a “modified” $\Sigma_1$ such that $\ker\{W_0 \Sigma_1 W_0^* \cap S^\perp \} = S^\perp$ and then employ the inequality

\[ \theta_k(\bar{x}_k(\sigma)) = \max \{0, 1 - \frac{d}{\|k\|} \} \]

for $k \in \mathcal{F}$.

Fig. 1. Behavior of $\theta$ for system $\Theta$ of Example 3.
$Z_k(V_0, \Sigma) \leq Z_k(V_0, \Sigma_1)$ of Proposition 1 (iv) to extend the result to the original $\Sigma$; see the proof of Theorem 1. Since $\Sigma_1$ does not excite $S^\perp$ and excites all $S$, there is no need to consider excited and nonexcited subspaces of $S^\perp$ and $S$ (as opposed, e.g., to $U \cap F$ and $U \cap F^\perp$). However, the above inequality is not suitable for dealing with the necessary condition for PS. That is why we study the projection $S$ in (i) of Lemma 9, and $T$ and $U$ in (ii) of the same lemma.

Example 4. Consider the systems $\Theta$ and $\Theta_X$ with

$$ (17) \quad A = \begin{bmatrix} 1 & -0.1 & 0 \\ 0.2 & 1 & 0 \\ 0 & 0 & 0.99 \end{bmatrix}, \quad \Sigma = \sigma \sigma', \quad \sigma = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. $$

Set $W_k = A^{-k}$, with

$$ \tilde{A} \approx \begin{bmatrix} 0.9901 & -0.099 & 0 \\ 0.1980 & 0.9901 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \tilde{A} = \begin{bmatrix} \sqrt{1.02} & 0 & 0 \\ 0 & \sqrt{1.02} & 0 \\ 0 & 0 & 0.99 \end{bmatrix}. $$

Figure 2 (i) illustrates how simple the behavior of $\bar{x}_k$ is when compared to $x_k$. The oscillatory behavior of $x_k$ prevents convergence of $\theta_V(x_k)$ for any fixed $V$. The hypothesis of Lemma 7 holds with $\ker(W_0\Sigma W_0') \cap U = [e_2]$ and, in fact, for $F = [e_1]$, $\theta_F \cap \tilde{U}(\bar{x}_k(\sigma)) \geq 1$, and $\theta_F(\bar{x}_k(\sigma)) \leq 0.3 \times 0.96^k$. We have checked that $H_k \bar{x}_k(\sigma) = 0$, $k \geq 0$, confirming Lemma 8. The statements of Lemma 9 (ii)–(iv) hold with $\delta = 0.001$, $\lambda = 1.031$, and $o(k) = 0.985^k$; Figure 2 (ii) and (iii) illustrate the behavior of the quantities $\|T(I - H_k)v\|$ and $\|H_kUv\|$ for $v = (\sqrt{3}/3) [1 \quad 1 \quad 1]'$. Note that $\|H_kUv\|$ presents an oscillation due to the fact that $H_k$ are nonorthogonal projections.

![Simulation results for system $\Theta_X$ of Example 4.](image)

**Fig. 2.** Simulation results for system $\Theta_X$ of Example 4. (i) State trajectory $\bar{x}_k(\sigma)$. (ii) and (iii) The quantities of Lemma 9 (ii) and (iv).

An important feature of the case with $\ker(W_0\Sigma W_0') \cap U \neq \{0\}$ is that $\text{Im}(\bar{H}) \cap U \neq \{0\}$, which follows from the fact that $\bar{H}$ cannot “cover” $U \ominus F$ as stated in Lemma 7. This fact, together with the structure of invariant spaces presented in Lemma 4, allows us to pick an initial condition $\bar{z}$ for which the associated $\bar{z}_k$ has a nontrivial projection onto $U \ominus F$, as stated in the next proposition, the proof of which is omitted.

**Proposition 4.** If $\ker(W_0\Sigma W_0') \cap U \neq \{0\}$, then there exists $\bar{z} \in S$ such that $U \bar{z}_k \neq 0$, $k \geq 0$. 

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4. Testable condition for PS and PSS. This section presents, initially, a sufficient condition for PS, with an extension to PSS. Then a necessary condition for PSS is presented and extended to PS. Finally, the results are gathered together in Theorem 1.

4.1. Sufficient conditions.

Lemma 10. Consider $W_0$ as in Proposition 2, the subspace $S$ as in (15), and $\tilde{z}_k$ as in (12). If $\ker \{W_0 \Sigma W_0' \} \cap S = \{0\}$ and $\ker \{W_0 \Sigma W_0' \} \cap S^\perp = S^\perp$, then for each $\tilde{z}$ there exist $\chi \geq 0$ and $0 \leq \beta < 1$ such that $\|\tilde{z}_k\| \leq \chi^{\beta k}$.

Proof. For ease of notation, in this proof we write $\tilde{z}$, $\tilde{A}$, and $\tilde{H}$ as $z$, $A$, and $H$, respectively; for $k, \ell \geq 0$, $w_{1,\ell}, w_{2,\ell}, w_{3,\ell}$ stand for vectors with $\|w_{j,\ell}\| \leq 1$. Recall the orthogonal projection $S$ used in Lemma 9. From Lemma 4 we have that both $S^\perp$ and $S$ are $A$-invariant, in such a manner that $AS_{k+\ell} \in S^\perp$ and $A(I - S)z_{k+\ell} \in S$, $k, \ell \geq 0$. Moreover, $\ker \{W_0 \Sigma W_0' \} \cap S = \{0\}$, and hence the conditions of Lemma 9 (i) hold, allowing us to evaluate, for $k, \ell \geq 0$,

$$
SH_{k+\ell+1}(AS_{k+\ell}) = S(A(Sz_{k+\ell}) + \pi \rho^k \ell + 1 \|A\| (w_{1,\ell}$$

$$= AS_{z_{k+\ell}} + \pi \rho^k \ell + 1 \|A\| (w_{1,\ell},

$$

$$H_{k+\ell+1}(I - S)z_{k+\ell} = H_{k+\ell+1}(I - S)(A(I - S)z_{k+\ell})$$

$$= \pi \rho^k \ell + 1 \|A\| (w_{2,\ell}$$

where $\pi, \rho$ are as in Lemma 9. Now we shall show inductively that

$$z_{k+\ell+1} = A^{\ell+1}S_{z_k} + (I - S)H_{k+\ell+1}A^{\ell+1}S_{z_k}$$

$$+ 2\pi \|A\|^{\ell+1} \|z_k\| (\rho^{k+1} + \cdots + \rho^k \ell + 1)w_{3,\ell}, \quad \ell \geq 0.$$

For $\ell = 0$, from (4) and (18) we have that

$$z_{k+1} = H_{k+1}A z_k = H_{k+1}A S z_k + H_{k+1}(I - S) z_k$$

$$= SH_{k+1}A S z_k + (I - S)H_{k+1}A S z_k + H_{k+1}(I - S) z_k$$

$$= AS_{z_k} + (I - S)H_{k+1}A S z_k$$

$$+ \pi \rho^k \ell + 1 \|A\| (w_{1,0} + \|A(I - S)z_k\| w_{2,0})$$

$$= AS_{z_k} + (I - S)H_{k+1}A S z_k + \pi \rho^k \ell + 1 (2\|A\| \|z_k\| w_{3,0}),$$

and assuming (19) holds for $\ell = 1$, similarly to the above we evaluate from (4)

$$z_{k+\ell+1} = H_{k+\ell+1}A z_{k+\ell}$$

$$= H_{k+\ell+1}A^{\ell+1}S_{z_k}$$

$$+ H_{k+\ell+1}(I - S)H_{k+\ell+1}A^\ell S_{z_k}$$

$$+ H_{k+\ell+1}(2\pi \|A\|^\ell \|z_k\| (\rho^{k+1} + \cdots + \rho^k \ell + 1)w_{3,\ell-1})$$

$$= SH_{k+\ell+1}A^\ell S_{z_k} + (I - S)H_{k+\ell+1}A^{\ell+1}S_{z_k}$$

$$+ H_{k+\ell+1}(I - S)H_{k+\ell+1}A^\ell S_{z_k}$$

$$+ H_{k+\ell+1}(2\pi \|A\|^\ell \|z_k\| (\rho^{k+1} + \cdots + \rho^k \ell + 1)w_{3,\ell-1})$$
and, from (18),

\[ z_{k+\ell+1} = A^{\ell+1}Sz_k + \pi \rho^{k+\ell+1}\|A^{\ell+1}Sz_k\|w_{1,k+\ell+1} + (I - S)H_{k+\ell+1}A^{\ell+1}Sz_k + \pi \rho^{k+\ell+1}\|A(I - S)H_{k+\ell+1}A^{\ell+1}Sz_k\|w_{2,k+\ell+1} + H_{k+\ell+1}A(2\pi\|A\|^{\ell+1}\|z_k\|(\rho^{k+1} + \cdots + \rho^{k+\ell})w_{3,\ell-1}) = A^{\ell+1}Sz_k + (I - S)H_{k+\ell+1}A^{\ell+1}Sz_k + 2\pi\|A\|^{\ell+1}\|z_k\|(\rho^{k+1} + \cdots + \rho^{k+\ell+1})w_{3,\ell}, \]

completing the inductive proof of (19). Then we can write, for \(k, \ell \geq 0\),

\[
\|z_{k+\ell+1}\| \leq \|A^{\ell+1}Sz_k\| + \|(I - S)H_{k+\ell+1}A^{\ell+1}Sz_k\| + 2\pi\|A\|^{\ell+1}\|z_k\|(\rho^{k+1} + \cdots + \rho^{k+\ell+1}) \leq 2\|A^{\ell+1}Sz_k\| + 2\pi\|A\|^{\ell+1}\|z_k\|(\rho^{k+1} + \cdots + \rho^{k+\ell+1}).
\]  

Now consider the term \(A^{\ell}Sz_k\), \(\ell \geq 1\). Since \(S^k\) is \(A\)-invariant and corresponds to the subspace spanned by eigenvectors associated to eigenvalues (strictly) inside the unit disk, one has that \(\|A^{\ell}Sz_k\| \leq \eta_\gamma\|Sz_k\| \leq \eta_\gamma\|z_k\|\) for some scalars \(\eta \geq 0\) and \(0 \leq \gamma < 1\). Then we set

\[ \ell_0 : \eta_\gamma\ell_0 \leq 1/4, \]

and from (20) with \(\ell = \ell_0 - 1\) we obtain

\[
\|z_{k+\ell_0}\| \leq (2\eta_\gamma\ell_0 + 2\pi\|A\|^{\ell_0}(\rho^{k+1} + \cdots + \rho^{k+\ell_0}))\|z_k\| \leq (1/2 + 2\pi\|A\|^{\ell_0}(\rho^{k+1} + \cdots + \rho^{k+\ell_0}))\|z_k\|, \ k \geq 0.
\]

Now we set \(k_0\) such that \(2\pi\|A\|^{\ell_0}(\rho^{k_0+1} + \cdots + \rho^{k_0+\ell_0}) < 1/2\). From (21) with \(k = k_0\), we obtain \(\|z_{k_0+\ell_0}\| \leq \bar{\beta}\|z_{k_0}\|\), where \(\bar{\beta} = (1/2 + 2\pi\|A\|^{\ell_0}(\rho^{k_0+1} + \cdots + \rho^{k_0+\ell_0})) < 1\); similarly, from (21) with \(k = k_0 + m\ell_0\), \(m \geq 0\), we obtain

\[
\|z_{k_0+m\ell_0+\ell_0}\| \leq \bar{\beta}\|z_{k_0+m\ell_0}\| \leq \bar{\beta}^2\|z_{k_0+(m-1)\ell_0}\| \leq \cdots \leq \bar{\beta}^{m+1}\|z_{k_0}\|.
\]

Finally, we have that each \(k \geq k_0\) can be written in the form \(k = k_0 + m\ell_0 + r\) for some \(0 \leq r < \ell_0\) and \(m\) with \((k - k_0)/\ell_0 \leq m \leq (k - k_0)/\ell_0 + 1\), leading to

\[
\|z_k\| \leq \|A\|^r\|z_{k_0+m\ell_0}\| \leq \|A\|^r\|z_{k_0}\|\bar{\beta}^m(\gamma^{1/\ell_0})^k, \ k \geq k_0,
\]

and since \(\|z_k\| \leq \|A\|^k\|z_0\|\), \(k < k_0\), it is a simple matter to check that we can set \(\beta = \beta^{1/\ell_0} < 1\) and find \(\chi \geq 0\) for which \(\|z_k\| \leq \chi\beta^k, \ k \geq 0\). \(\square\)

Lemma 10 can be easily extended to the context of semistability of the system \(\Theta\) by employing \(\xi < 1\) as a scaling factor that “converts” \(U\) associated with the matrix \(\tilde{A}\) into \(S\xi\) associated with \(\xi\tilde{A}\).

**Corollary 1.** Consider the system \(\Theta\), \(W_0\) as in Proposition 2 and \(U\) as in (15). If \(\ker\{W_0\Sigma W_0^T\} \cap U = \{0\}\) and \(\ker\{W_0\Sigma W_0^T\} \cap U^+ = U^+\), then for each \(z\) and \(0 \leq \zeta < 1\) there exist \(\alpha \geq 0\) and \(0 \leq \beta < 1\) such that \(\|\zeta^{k}z_k\| \leq \alpha\beta^k\).
Proof. Let \( \Theta_Z, S, \) and \( U \) correspond to the matrix \( A \) and, for \( 0 \leq \xi < 1 \), let \( \Theta_Z, S, \) and \( U \) correspond to the matrix \( \xi A \). Note that the eigenvalues of \( \xi A \) lying in the unit disk are shifted to eigenvalues of \( \xi \widehat{A} \) inside the disk, yielding \( U \supset S \) for a general \( 0 \leq \xi < 1 \). Let \( \xi \) be sufficiently close to one, in such a manner that \( \xi \geq \zeta \) and \( U = S \). This leads to \( \{ \ker\{W_0\Sigma W_0'\} \cap S \xi \} = \{ \ker\{W_0\Sigma W_0'\} \cap U \} = \{ 0 \} \); furthermore, \( S_{\xi} = U \perp \) yields \( \{ \ker\{W_0\Sigma W_0'\} \cap S_{\xi} \} = \{ \ker\{W_0\Sigma W_0'\} \cap U \perp \} = U \perp S_{\xi} \). Employing the result of Lemma 10 for system \( \Theta_Z \) yields that there exist \( \chi \geq 0 \) and \( 0 \leq \beta < 1 \) for which

\[
\| \xi k \| \leq \chi \beta^k. 
\]

From Proposition 3 we get that \( \xi k = \xi k \xi k \), allowing us to obtain from (22)

\[
\| \xi k \| \leq \chi \beta^k. 
\]

and Lemma 3 (ii) provides \( \| \xi k \| \leq \chi \beta^k \). Recalling that \( \zeta \leq \xi \), we have \( \| \xi k \| \leq \| \xi k \| \leq (1 - \kappa)^{-1} \chi \beta^k \).

4.2. Necessary conditions. Conversely to Corollary 1, if \( \Sigma \) does not completely excite \( U \), then exponential divergence takes place. It is convenient, for later reference, to formalize the result as follows.

**Lemma 11.** Consider the system \( \Theta_Z, W_0 \) as in Proposition 2 and \( U \) as in (15). If \( \ker\{W_0\Sigma W_0'\} \cap U \neq \{ 0 \} \), then there exist \( z \in \mathbb{R}^n \) and \( 0 \leq \zeta < 1 \) such that for all \( \chi \geq 0 \) and \( 0 \leq \psi < 1 \), \( \| \psi k \| \leq \chi \psi^k \) for some \( k \geq 0 \).

**Proof.** We start setting \( \zeta < 1 \) sufficiently close to one, in such a manner that \( \zeta(1 + \delta) > 1 \), where \( \delta \) is as in Lemma 9, and, simultaneously, \( \lambda_i(\xi A) \notin \mathbb{D} \) if and only if \( \lambda_i(\xi A) \notin \mathbb{D} \), \( 0 < i \leq n \) (the unstable space of \( \xi A \) equals the unstable space of \( A \)). For ease of notation, in what follows we write \( \xi, \widehat{A}, \) and \( \bar{H} \) as \( z, A, \) and \( H \), respectively; for \( \ell \geq 0 \), \( w_{1,\ell}, w_{2,\ell}, w_{3,\ell} \) stand for vectors with \( \| w_{j,\ell} \| \leq 1 \). We shall need an evaluation that is analogous to (19) of Lemma 10. In fact, (19) involves projections onto \( S_{\perp} \) and \( S \) via \( S \) and \( I - S \), respectively, and now we consider projections onto \( U \perp F \), \( F \), and \( U \perp F \) via \( U \), \( (I - T) \), and \( (I - U)T \), respectively. Using Lemma 9 (ii) and (iii) yields

\[
z_{k+1} = (TA)^{\ell + 1} U z_k + A^{\ell + 1} (I - U) T z_k \\
+ (I - T) H_{k+\ell + 1} (TA)^{\ell + 1} U z_k \\
+ (I - T) H_{k+\ell + 1} A^{\ell + 1} (I - U) T z_k \\
+ 4 \| A \|^{\ell + 1} \| z_k \| (o(k + 1) + \cdots + o(k + \ell + 1)) w_{3,\ell}, \ \ell \geq 0,
\]

and, since Lemma 9 (v) provides \( (TA)^{\ell + 1} U = (UA)^{\ell + 1} U \), this can be written as

\[
z_{k+1} = (UA)^{\ell + 1} U z_k + A^{\ell + 1} (I - U) T z_k \\
+ (I - T) H_{k+\ell + 1} (UA)^{\ell + 1} U z_k \\
+ (I - T) H_{k+\ell + 1} A^{\ell + 1} (I - U) T z_k \\
+ 4 \| A \|^{\ell + 1} \| z_k \| (o(k + 1) + \cdots + o(k + \ell + 1)) w_{3,\ell}, \ \ell \geq 0.
\]
which can be substituted in the third term on the right-hand side of (24), leading to
\begin{equation}
\begin{aligned}
    z_{k+\ell+1} &= H_{k+\ell+1}(UA)^{\ell+1}Uz_k - o(k + \ell + 1)\|A\|^{\ell+1}Uz_k \|w_{1,\ell} \\
    &+ A^{\ell+1}(I - U)Tz_k + (I - T)H_{k+\ell+1}A^{\ell+1}(I - U)Tz_k \\
    &+ 4\|A\|^\ell z_k \| (o(k + 1) + \cdots + o(k + \ell + 1))w_{3,\ell} \\
    &+ 5\|A\|w_{\ell+1}z_k \| (o(k + 1) + \cdots + o(k + \ell + 1))w_{4,\ell}, \quad \ell \geq 0.
\end{aligned}
\end{equation}

(25)

Regarding the second and third terms on the right-hand side of (25), recall that 
\((I - U)Tz \in (U^\perp \cap F) \subset U^\perp\), yielding that \(A^\ell(I - U)Tz\) may present polynomial divergence, as \(k \to \infty\); hence we can write for each \(\gamma > 1\), and in particular for \(\gamma\) such that \(\gamma \ell < 1\),
\begin{equation}
\begin{aligned}
    \|A^{\ell+1}(I - U)Tz_k + (I - T)H_{k+\ell+1}A^{\ell+1}(I - U)Tz_k\| \\
    \leq \|A^{\ell+1}(I - U)Tz_k\| + \|(I - T)\|H_{k+\ell+1}\|A^{\ell+1}(I - U)Tz_k\| \\
    = 2\|A^{\ell+1}(I - U)Tz_k\| \leq 2\eta\gamma^{\ell+1}\|z_k\|
\end{aligned}
\end{equation}

for some \(\eta \geq 0\). Note that (25), (26), and Lemma 9 (v) lead to
\begin{equation}
\begin{aligned}
    \|z_{k+\ell+1}\| \leq \|H_{k+\ell+1}(UA)^{\ell+1}Uz_k\| + 2\eta\gamma^{\ell+1}\|z_k\| \\
    &+ 5\|A\|^\ell z_k \| (o(k + 1) + \cdots + o(k + \ell + 1)) \\
    \leq \lambda\|(UA)^{\ell+1}Uz_k\| + 2\eta\gamma^{\ell+1}\|z_k\| \\
    &+ 5\|A\|w_{\ell+1}z_k \| (o(k + 1) + \cdots + o(k + \ell + 1)).
\end{aligned}
\end{equation}

(27)

On the other hand, premultiplying both sides of (24) by \(U\) and employing the fact that, for \(v \in \mathbb{R}^n\), \((I - T)v \in F\), yielding \((I - T)v \perp (F^\perp \cap U)\) and hence \(U(I - T) = 0\), evaluations similar to the above ones provide
\begin{equation}
\begin{aligned}
    \|Uz_{k+\ell+1}\| = \|(UA)^{\ell+1}Uz_k + UA^{\ell+1}(I - U)Tz_k \\
    &+ 4\|A\|^\ell z_k \| (o(k + 1) + \cdots + o(k + \ell + 1))Uw_{3,\ell}\| \\
    \geq \|(UA)^{\ell+1}Uz_k\| - \eta\gamma^{\ell+1}\|z_k\| \\
    &- 4\|A\|w_{\ell+1}z_k \| (o(k + 1) + \cdots + o(k + \ell + 1)).
\end{aligned}
\end{equation}

(28)

By substituting (28) in (27) we get that
\begin{equation}
\begin{aligned}
    \|z_{k+\ell+1}\| \leq \lambda\|(Uz_{k+\ell+1}\| + \eta\gamma^{\ell+1}\|z_k\| \\
    &+ 4\|A\|^\ell z_k \| (o(k + 1) + \cdots + o(k + \ell + 1)) \\
    &+ 2\eta\gamma^{\ell+1}\|z_k\| + 5\|A\|w_{\ell+1}z_k \| (o(k + 1) + \cdots + o(k + \ell + 1)) \\
    = \lambda\|Uz_{k+\ell+1}\| + (2 + \lambda)\eta\gamma^{\ell+1}\|z_k\| \\
    &+ (5 + 4\lambda)\|A\|w_{\ell+1}z_k \| (o(k + 1) + \cdots + o(k + \ell + 1)), \quad \ell \geq 0,
\end{aligned}
\end{equation}

or equivalently, for \(\ell \geq 0\),
Recalling that \( (U \ominus F) \) is associated with unstable eigenvalues of \( A \). We proceed similarly to the above, employing (24) with \( k \) replaced by \( k + \ell + 1 \) and \( \ell \) replaced by \( m - 1 \), and Lemma 9 (ii)–(v), to evaluate

\[
\| z_{k+\ell+m+1} \| \geq \|(UA)^mUz_{k+\ell+1} + (I - T)H_{k+\ell+m+1}(UA)^mUz_{k+\ell+1} + (I - T)H_{k+\ell+m+1}A^m(I - U)Tz_{k+\ell+1}\|
\]

\[
- 4\|A^m\|z_{k+\ell+1}\|(o(k + \ell + 2) + \cdots + o(k + \ell + m + 1))
\]

Recalling that \( (I - T) \) and \( U \) are orthogonal projections and employing Lemma 9 (iii) and an evaluation similar to (26) (with the same \( \gamma \) as in (26), such that \( \gamma \zeta < 1 \), the above inequality leads to

\[
\| z_{k+\ell+m+1} \| \geq \|((UA)^mUz_{k+\ell+1} - \|A^m(I - U)Tz_{k+\ell+1}\|
\]

\[
- 4\|A^m\|z_{k+\ell+1}\|(o(k + \ell + 2) + \cdots + o(k + \ell + m + 1))
\]

and, from (29),

\[
\| z_{k+\ell+m+1} \| \geq (1 + \delta)^m\lambda^{-1}\left(\| z_{k+\ell+1} \| - (2 + \lambda)\eta\gamma^{\ell+1}\|z_k\|
\]

\[
- (5 + 4\lambda)\|A\|^\ell+1(o(k + 1) + \cdots + o(k + \ell + 1))\|z_k\|
\]

\[
- \eta\gamma^m\|z_{k+\ell+1}\|
\]

\[
- 4\|A^m\|(o(k + \ell + 2) + \cdots + o(k + \ell + m + 1))\|z_{k+\ell+1}\|
\]

Multiplying both sides of the above equation by \( \zeta^{k+\ell+m+1} \) and setting \( \xi = \gamma \zeta \) and \( z_{\xi,k} = \zeta^k z_k, \ell \geq 0 \), lead to

\[
\| z_{\xi,k+\ell+m+1} \| \geq \xi^m(1 + \delta)^m\lambda^{-1}\left(\| z_{\xi,k+\ell+1} \| - (2 + \lambda)\eta\xi^{\ell+1}\|z_{\xi,k}\|
\]

\[
- (5 + 4\lambda)\xi^{\ell+1}(o(k + 1) + \cdots + o(k + \ell + 1))\|z_{\xi,k}\|
\]

\[
- \eta\xi^m\|z_{\xi,k+\ell+1}\|
\]

\[
- 4\xi^m\|A\|^m(o(k + \ell + 2) + \cdots + o(k + \ell + m + 1))\|z_{\xi,k+\ell+1}\|
\]

Recall that \( \xi(1 + \delta) > 1, \xi = \gamma \zeta < 1 \), and \( o(\cdot) \) is a decreasing function, allowing for the following settings. For an arbitrary \( \bar{m} > 0 \), let \( m \) be such that

\[
\xi^m(1 + \delta)^m\lambda^{-1} - \eta\xi^m > 6\bar{m}.
\]

Set \( \ell = \max(\ell_1, \ell_2) \), where \( \ell_1 \) is such that

\[
\xi^m(1 + \delta)^m\lambda^{-1} - \eta\xi^m < (1/2)\bar{m}
\]
Finally, let $\ell_2$ be such that
\begin{equation}
4\zeta^m \|A\|^m (o(\ell_2 + 2) + \cdots + o(\ell_2 + m + 1)) < 3\bar{m}, \quad k \geq 0.
\end{equation}

Substituting (31)–(34) in (30) provides
\begin{equation}
\|z_{k+k+\ell+1+m}\| \geq 6\bar{m}\|z_{k+\ell+1}\| - (1/2)\bar{m}\|z_{k+1}\| - (1/2)\bar{m}\|z_{k}\| - 3\bar{m}\|z_{k+\ell+1}\| = \bar{m}\|z_{k+\ell+1}\| + 2\bar{m}\|z_{k+\ell+1}\| - \bar{m}\|z_{k}\|.
\end{equation}

Using this inequality in a recursive fashion, substituting $k$ with $k + qm$, $q \geq 0$, we obtain
\begin{equation}
\|z_{k+\ell+1+(q+1)m}\| \geq \bar{m}q\|z_{k+\ell+1+m}\| + \sum_{j=0}^{q} (2\bar{m})^j \|z_{k+\ell+1+jm}\| - \sum_{j=0}^{q} \bar{m}^j \|z_{k+jm}\|.
\end{equation}

The second term on the right-hand side of (35) dominates the third one, leading to exponential divergence; in particular, we can write for $q \geq q_0$ that
\begin{equation}
\|z_{k+\ell+1+(q+1)m}\| \geq \|z_{k+\ell+1+m}\|, \quad \text{leading to}
\end{equation}
\begin{equation}
\zeta^{k+\ell+1+(q+1)m} \|z_{k+\ell+1+(q+1)m}\| \geq \zeta^{k+\ell+1+m} \|z_{k+\ell+1+m}\|
\end{equation}
or, equivalently, \(\|z_{k+\ell+1+(q+1)m}\| \geq \zeta^{-q}\|z_{k+\ell+1+m}\|, q \geq q_0\). It is important to mention that we can pick an initial condition \(z\) for which \(z_{k+\ell+1+m} \neq 0\); see Proposition 4.

Similarly to the proof of Corollary 1, we employ a “scaling factor” \(\xi < 1\) that converts \(S\) associated with \(A\) into \(U_{\xi}\), associated with \(\xi^{-1}A\), and extend the result of Lemma 11 to the context of PS. It is worth mentioning that Lemma 11 applied to matrix \(\xi^{-1}A\) provides \(\|\xi^{k}\xi^{-k}z_k\| > \chi \psi^k\) (in analogy with (23)), or, equivalently, \(\|z_k\| > \chi (\psi^{-1}\xi)^k\), and we can set both \(\zeta\) and \(\xi\) arbitrarily close to one, to obtain the next result, the proof of which is not presented.

**Corollary 2.** Consider the system \(\Theta\), \(W_0\) as in Proposition 2 and \(S\) as in (15). If \(\ker\{W_0 \Sigma W_0'\} \cap S \neq \{0\}\), then there exist \(z \in \mathbb{R}^n\) such that for all \(\chi \geq 0\) and \(0 \leq \psi < 1\), \(\|z_k\| > \chi \psi^k\) for some \(k \geq 0\).

**4.3. Main result.**

**Theorem 1.** Consider the system \(\Theta\). Let \(J\) represent the similarity transformation for which \(JAJ^{-1}\) is in Jordan form and let \(J\) and \(J_S\) stand for the unstable space and the stable space of \(JAJ^{-1}\), respectively. (\(A, \Sigma\)) is PS if and only if
\begin{equation}
\ker\{J\Sigma J'\} \cap J = \{0\}.
\end{equation}

\((A, \Sigma)\) is PSS if and only if
\begin{equation}
\ker\{J\Sigma J'\} \cap J_S = \{0\}.
\end{equation}

**Proof.** Regarding the first statement, consider \(W_0\) as in Proposition 2 and \(U\) as in (15). Condition (36) holds if and only if
\begin{equation}
\ker\{W_0 \Sigma W_0'\} \cap U = \{0\}.
\end{equation}
see Lemma 5. Assume that \( \ker\{W_0\Sigma W_0'\} \cap S^\perp = S^\perp \). It follows from Corollary 1 and Lemma 11 that (38) is a necessary and sufficient condition for the existence, for each \( z \in \mathbb{R}^n \) and \( 0 \leq \zeta < 1 \), of \( \alpha \geq 0 \) and \( 0 \leq \beta < 1 \) such that \( \|x^k z_k\| \leq \alpha \beta^k \). Lemma 2 extends the result to PSS of \((A, \Sigma)\). Now we consider the case when \( \ker\{W_0\Sigma W_0'\} \cap S^\perp \neq S^\perp \), that is, \( \Sigma \) also excites \( S^\perp \). In this situation, we can write \( \Sigma = \Sigma_1 + \Sigma_2 \) with \( \Sigma_1, \Sigma_2 \in \mathbb{R}^{n_0} \) and \( \ker\{W_0\Sigma_1 W_0'\} \cap S^\perp = S^\perp \), to conclude that \((A, \Sigma_1)\) is PSS, that is, for each \( 0 \leq \zeta < 1 \) and \( V \in \mathbb{R}^{n_0} \), there exists \( \tilde{Z} \in \mathbb{R}^{n_0} \) for which

\[
Z_k(V, \Sigma_1) \leq \zeta^{-k} \tilde{Z}.
\]

Finally, since \( \Sigma - \Sigma_1 = \Sigma_2 \geq 0 \), Proposition 1 (iv) yields \( Z_k(V, \Sigma) \leq Z_k(V, \Sigma_1) \) and (39) provides \( Z_k(V, \Sigma) \leq \zeta^{-k} \tilde{Z} \), i.e., \((A, \Sigma)\) is PSS. The proof for the second statement follows from Corollary 2 and Lemma 10 in a similar manner as above.

Remark 2. Either \( \Sigma > 0 \) or semistable \( A \) imply \((A, \Sigma)\) is PSS, which implies that \((A, \Sigma)\) is semistabilizable. Indeed, \( \Sigma > 0 \) provides \( \ker\{J\Sigma J'\} = \{0\} \) and semistable \( A \) yields \( J = \{0\} \), and in both cases (36) holds. Regarding the second implication, \((A, \Sigma)\) not semistabilizable means that \( \Sigma \) does not excite an “entire” unstable mode of \( A \), and (36) does not hold. PSS is not comparable to stabilizability of \((A, \Sigma)\); indeed, Example 6 illustrates the situation when \((A, \Sigma)\) is stabilizable but \( \Theta \) is not PSS, whereas system \( \Theta \) with \( A = 1 \) and \( \Sigma = 0 \) illustrates the opposite situation. Similarly, stable \( A \) imply that \((A, \Sigma)\) is PS, which implies that \((A, \Sigma)\) is stabilizable.

Example 5. Consider the system \( \Theta \) of Example 3 with \( |d| > 1 \). One has that \( J = I \), \( \ker\{J\Sigma J'\} = \{e_1\} \), and \( J = J^\perp = \mathbb{R}^n \). Neither (36) nor (37) holds, and from Theorem 1 we have that \((A, \Sigma)\) is not PS or PSS. For \( |d| < 1 \), \( J = J^\perp = \{0\} \), and the system is PS and PSS. For \( |d| = 1 \), \( J = \{0\} \) and \( J^\perp = \mathbb{R}^n \), and hence only (36) holds and the system is “strictly” PSS; the same holds if \( \sigma = e_2 \) is replaced with \( \sigma_2 = 0 \), as in Example 1.

Example 6. Consider the system \( \Theta \) of Example 4. It is simple to check that (36) is not satisfied and, according to Theorem 1, \((A, \Sigma)\) is not PSS. See Figure 3 (i) for the behavior of the \( Z \)-component of the state trajectory. Note that \((A, \Sigma)\) is stabilizable but the system is not PSS. Now, replace \( A \) with \( 1.02^{-1/2}A_0 \), which leads to \( \lambda(A) \in \overline{D} \) and \( J = \{0\} \). In this situation, only (36) holds. Note via Figure 3 (ii) that the \( Z \)-component presents an oscillatory behavior, which is compatible with PSS.

![Fig. 3. Behavior of the Z-component for the setups of Example 6.](image)

Remark 3. The conditions of Theorem 1 can be employed in the problem of “stabilization” of the system \( \Theta \) via a suitable choice of \( \Sigma \). For instance, \( \Sigma = \sigma_1 \sigma_1' + \cdots + \sigma_r \sigma_r' \) with \( r = \dim(J) \) is such that \((A, \Sigma)\) is PSS if and only if \( \sigma_j, 0 \leq j \leq r, \)
are linearly independent vectors with nontrivial projections onto $\mathcal{J}$, and a similar condition holds for PS. As illustration, for the system $\Theta$ of Example 5, if we set $\sigma_1 = e_1$ and $\sigma_2 = e_2$, we obtain both PS and PSS; the same is valid for the system of Example 6 (of course, now $e_1, e_2 \in \mathbb{R}^3$ and $\Sigma$ is rank deficient).

5. Concluding remarks. In this paper we have explored the structure of the system $\Theta$ in (1), with special attention to the relations among the initial condition $\Sigma$ of the $X$-component, its dynamics (governed by $A$), and the coupling with the $Z$-component via the orthogonal projection $H$. We obtain the structural, testable condition (36) for PSS, with the interpretation that $\Sigma$ has to completely excite the unstable modes of $A$. Similarly, the condition (37) for PS requires that $\Sigma$ excite all modes of $A$ except the stable ones. These interpretations are particularly meaningful in the scenario of Kalman filtering for linear time-invariant systems, meaning that the noise in the initial condition excites the unstable or the “semiunstable” dynamics of the plant; indeed, the derived conditions are essential to obtain, as discussed in [4], necessary and sufficient conditions for avoiding actual exponential divergence of estimates and bounded error estimates, which is a significant result, taking into account the conservativeness of existing conditions. The results can also be employed in the problem of stabilization of the filter via a suitable choice of noise model; see Remark 3.

REFERENCES