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THE VISCOSITY PARAMETER’S DEPENDENCE ON THE PROFILE OF THE DISK SCALE HEIGHT

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ABSTRACT

It is shown that, for accretion disks, the height scale is a constant whenever hydrostatic equilibrium and the subsonic turbulence regime hold in the disk. In order to have a variable height scale, processes are needed that contribute an extra term to the continuity equation. This contribution makes the viscosity parameter much greater in the outer region and much smaller in the inner region. Under these circumstances, turbulence is the presumable source of viscosity in the disk.

Subject headings: accretion, accretion disks — hydrodynamics — radiative transfer — stars: dwarf novae — turbulence

1. INTRODUCTION

In the past 35 years, accretion disks have become one of the most intense research areas in theoretical astrophysics. The reason for this cannot be explained just by invoking their ubiquity in a variety of different astrophysical environments. One should recognize the role played by the Shakura & Sunyaev (1973) α-standard model, which has the appeal of a very intuitive, simple, and neat physics. This model’s main assumptions about an accretion disk are that it has a geometrical thinness and a large optical thickness, that it is in hydrostatic equilibrium in z and is perpendicular to the plane of the disk, and that its approach to the viscosity parameterization is in terms of an unknown α-parameter. This model was meant to be applied to very specific conditions, such as those occurring in the outer parts of accretion disks in binary systems, around very young stellar objects, and in flows associated with the central engine of active galactic nuclei (Papaloizou & Lin 1995).

Soon, however, the astronomical and astrophysical community realized that the approach used to treat viscosity was a general one and that it could be used under quite different conditions. Somehow, this approach has encouraged the astronomical and astrophysical community to tackle related problems, without even knowing the source of the viscosity. This concern with the viscosity problem has gained tremendous momentum with the work of Balbus & Hawley (1991), who rediscovered the work of Velikhov (1959) and Chandrasekhar (1960) on the magnetorotational instability of Couette flows, which they propose as the process of viscosity generation in accretion disks.

However, some points related to the interpretation of theoretical simulations, as well as their application to real systems, must still be understood and clarified (King et al. 2007). The level of turbulence this process may sustain is still a matter of debate, yielding a rather large range of α. Progress in understanding accretion disks by the use of this process has been huge but has not yet reached the stage of allowing comparison with observations. Perhaps this may explain why some astronomers would rather obtain α from their own data and keep using the α Ansatz.

The applications were so successful and profuse that, today, one speaks of at least four models of accretion disks: (1) the α-standard model, which is successfully applied to cataclysmic variables, transient X-ray sources, accretion disks around active galactic nuclei, and accretion disks in young stellar objects (Smak 1999; Lasota 2001; Menou et al. 2000; King et al. 2007; Begelman 1985; Lin 1989); (2) the advection-dominated accretion flow model, which is used to explain X-ray and γ-ray emission from underluminous X-ray binaries and active galactic nuclei (Perna et al. 2000; Narayan & Yi 1995; Becker & Le 2003; Narayan et al. 2002); (3) the convection-dominated accretion flow model, which is applied to underfed and radiatively inefficient hard X-ray binaries and active galactic nuclei (Abramowicz et al. 2002; Narayan et al. 2000; Igumenshchev 2002; Igumenshchev et al. 2003; Narayan et al. 2002; Quataert & Gruzinov 2000; Ball et al. 2001; Yuan et al. 2003); and (4) the Shapiro et al. (1976) two-temperature accretion disk model, which is applied to Cygnus X-1.

Despite the remarkable differences between these models and the systems to which they apply, the α-parameterization works quite well. One of the reasons for this is the weak dependence on α of the disk properties (King et al. 2007). Besides, it seems that the range of values of α that fit all these systems is not large, and, consequently, these fairly similar values of α result in reasonable agreement with observations (King et al. 2007).

However, there are some points that require more detailed treatment, and these are related to the disk scale height. By this, we mean not only its value but, above all, its behavior along the disk. These questions are of fundamental importance when one is concerned with characteristic timescale lengths, turbulence regimes, and the criteria to choose among different energy transport models.

To make this point more clear, we shall focus on the α-standard model, but our criticism applies to them all. The assumed constancy of the viscosity parameter is very decisive in the α-standard model, leading to a disk scale height with a radial distance $r^{9/4}$, and yields the same behavior along r for both the effective temperature and the temperature at the midplane ($z = 0$) of the disk, $T \approx r^{-3/4}$. As a matter of fact, it leads to a constant optical depth along r. However, this result, which is apparently consistent with the constancy of the viscosity parameter, is made possible by an unsound interpretation of a formal solution to the continuity equation. Using the same assumptions as those found in the α-standard model, but with the constancy of the viscosity parameter, and correcting the solution to the continuity equation, we can change the results quite significantly. Now, α will go as $r^{11/25}$, with a huge variation along the disk. If the constancy of α is indeed required,
then one will have to look for some process that can add a term to the continuity equation so as to give the correct disk scale height.

It is our claim in this Letter that the questions we have just raised cast some doubts on the way in which the disk scale height is obtained and the way in which \(\alpha\) is determined. The results that we have obtained in this Letter modify previous results in the area and modify our current knowledge about the fundamental issues of accretion disk theory. They are also very important as far as full consistency is required but, thus far, have not been considered.

2. THE CONTINUITY EQUATION, RADIAL HEIGHT SCALE PROFILE, AND VISCOSITY PARAMETER

In this section, we show how the radial scale height is related to the mass continuity equation. We also show that, in order to have a disk scale height that varies with radial distance, one needs to look for processes that add an extra term to the continuity equation. Because the disk is assumed to have azimuthal symmetry, being under hydrostatic equilibrium in the \(\phi\)-direction, the continuity equation reads

\[
\frac{1}{r} \frac{\partial}{\partial r} (r \rho V_r) = 0, \tag{1}
\]

where \(r\), \(\rho\), and \(V_r\) are the radial distance, mass density, and radial velocity, respectively. It should be said that the above equation neglects mass transport due to turbulence, which is equivalent to assuming a subsonic turbulence regime. Now, we set \(f(r, z) = r \rho V_r\) and expand \(f\) in powers of \(z\) to obtain

\[
f(r, z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \left( \frac{\partial^{2n} f(r, z)}{\partial z^{2n}} \right)_{z=0}, \tag{2}
\]

where the reflexion symmetry over \(z\) is taken into account. Inserting \(f(r, z)\), which is given in equation (2), into equation (1) gives

\[
\frac{1}{r} \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \frac{\partial}{\partial r} \left( \frac{\partial^{2n} f(r, z)}{\partial z^{2n}} \right)_{z=0} = 0. \tag{3}
\]

Setting \(z = 0\), we must have

\[
\frac{\partial}{\partial r} f(r, 0) = 0, \tag{4}
\]

or

\[
f(r, 0) = C_0, \tag{5}
\]

where \(C_0\) is constant. For \(z \neq 0\), we recall that

\[
\frac{\partial}{\partial r} \left( \frac{\partial^{2n} f(r, z)}{\partial z^{2n}} \right)_{z=0} = \left( \frac{\partial^{2n+1} f(r, z)}{\partial z^{2n+1}} \right)_{z=0}, \tag{6}
\]

which, by use of equation (1), should give

\[
\frac{\partial^{2n+1}}{\partial z^{2n+1}} f(r, z) = 0. \tag{7}
\]

Then

\[
\left( \frac{\partial^{2n}}{\partial z^{2n}} f(r, z) \right)_{z=0} = \text{constant} = C_{2n}, \tag{8}
\]

and the most general solution to equation (1) should be written as

\[
f(r, x) = C_0 \sum_{n=0}^{\infty} \frac{C_{2n}}{C_0 (2n)!} x^{2n}, \tag{9}
\]

where \(x = z/\ell\) and \(C_0 = (r \rho V_r)_{z=0}\). Finally, for the accretion rate, we can write

\[
\dot{M} = 4\pi \ell C_0 \sum_{n=0}^{\infty} \frac{C_{2n}}{C_0 (2n + 1)!}, \tag{10}
\]

and because

\[
\frac{\partial}{\partial r} \dot{M} = 0, \tag{11}
\]

this implies

\[
\frac{\partial}{\partial r} \ell = 0, \tag{12}
\]

because the \(C_{2n}\) are all constants. We then must conclude that, under hydrostatic equilibrium, together with subsonic turbulence, the height scale of the disk is not allowed to vary along the radial distance \(r\). Now we are going to highlight some of the consequences of the conclusions that we have just drawn. In order to proceed, let us recall some very familiar results from the accretion disks theory. Let us start from the hydrostatic equilibrium equation (from which we define the disk height scale \(\ell\)):

\[
P = \rho \Omega^2 \ell^2, \tag{13}
\]

where \(P\) is the pressure and \(\Omega\) is the Keplerian angular velocity. From the angular momentum conservation equation, we obtain

\[
\rho = \frac{\dot{M}}{2\pi \Omega \ell^2} S, \tag{14}
\]

where \(\alpha\) is the viscosity parameter and \(S = 1 - (r_i/r)^{1/2}\) takes into account the null boundary condition for the torque at \(r = r_i\), which is the inner radius of the disk. Assuming the disk to be cooled by radiative transport in the \(z\)-direction, in the diffusion approximation, we can write

\[
F_z = -c \frac{\partial}{\partial z} P, \tag{15}
\]

where \(c\), \(F_z\), \(\tau\), and \(P\) are the velocity of light, radiative flux in the \(z\)-direction, optical depth, and radiation pressure, respec-
tively. Replacing differentials by finite differences, and recalling the definition of effective temperature, i.e.,

$$\sigma T_{\text{eff}}^4 = \frac{3}{4\pi} M \Omega^2,$$  \hspace{1cm} (16)

equation (16) may be written as

$$\frac{3}{4\pi \sigma} M \Omega^2 \tau = T^4,$$  \hspace{1cm} (17)

where $\sigma$ is the Stefan-Boltzmann constant and $T$ is the temperature at the midplane of the disk. In the outer parts of the disk, the opacity is mainly given by the free-free opacity. Therefore, using a Rosseland mean opacity, averaged over $z$, we can rewrite equation (17) as

$$T^4 = 2.62 \times 10^{-2} M \Omega^2 \rho^{-2} T^{-1.5} \ell.$$  \hspace{1cm} (18)

Finally, for a gas pressure–dominated disk, we obtain, from equation (18),

$$\alpha^2 = 1.33 \times 10^{-51} \frac{M_{17}}{M_{34}} \frac{\chi_{2.5}}{\chi_{0.125}},$$  \hspace{1cm} (19)

where $M_{17}$, $M_{34}$, $x$, and $y$ are the accretion rate in units of $10^{17}$ g s$^{-1}$, the mass of the central object in units of $10^{34}$ g, the radial distance in units of the inner radius, and the disk scale height in units of the inner radius. The inner radius $r_1$ is assumed to be $3R_s$ (with $R_s$ being the gravitational radius). According to King et al. (2007), if constraints on $\alpha$ are required, then one should try to observe only systems subject to temporal behavior, such as dwarf nova outbursts (Warner 2003; Cannizzo 2001), outbursts of X-ray transients (Lasota 2001), protostellar accretion disks (Hartmann et al. 1998), FU Orionis outbursts (Lodato & Clarke 2004), or optical variability in active galactic nuclei (Starling et al. 2004). For instance, if we take the viscous timescale

$$t = -\int_{r_1}^{r_0} \frac{r}{\nu} \, dr,$$  \hspace{1cm} (20)

where $\nu$ is the kinematic viscosity, then, by use of equation (19) and a little algebra, we obtain, for the standard model,

$$\frac{\ell}{r} = \frac{y}{x} = 1.63 \times 10^{-3} \frac{M_{17}}{M_{34}} \frac{\chi_{2.5}}{\chi_{0.125}}^{0.319} I_{0.25} x_{1.25},$$  \hspace{1cm} (21)

where $x_d$ is the disk size in units of the inner radius. Inserting this expression into equation (19) yields

$$\alpha_\ell = 0.1x_d^{-1.125} \alpha_0,$$  \hspace{1cm} (22)

or, compared with $t$ in Smak (1999),

$$t = 71.39 M_{34}^{0.3} M_{17}^{-0.8} \alpha_\ell^{-0.8} x_d^{1.25}.$$  \hspace{1cm} (23)

It should be noted that, contrary to Smak (1999), we use the disk scale height dependence on $T$, which is the temperature at $z = 0$, not $T_{\text{eff}}$; we believe that this is the correct procedure. Essentially, this procedure makes our scale height a factor of $r^{0.125}$ greater than that found in Smak (1999), with $\tau$ being the optical depth. Our results differ from Smak’s (1999) by minor discrepancies in the exponents but by a numerical factor of approximately $\geq 2$ orders of magnitude. Finally, using our formulation, we obtain

$$\frac{\ell}{r} = \frac{y}{x} = 2.1 \times 10^{-3} \left( \frac{M_{17}}{M_{34}} \right)^{0.1875} I_{0.125} x^{-1},$$  \hspace{1cm} (24)

and, for the value of $\alpha$ directly related to the viscous time,

$$\alpha_\ell = 0.1x_d^{-1.125} \alpha_0,$$  \hspace{1cm} (25)

with $\alpha_x$ given by equation (22). Equation (25) is the value of $\alpha$ at $x = 1$. The value of $\alpha$ anywhere in the disk is

$$\alpha = x^{1.125} \alpha_0.$$  \hspace{1cm} (26)

From equations (22) and (25), we see that the viscosity parameter related to the viscous time will be much smaller when we employ the correct solution to the continuity equation. At the outer radius of the disk, it will be $0.1x_d^{9.675}$, which is greater than the value obtained with the standard model.

### 3. Analysis and Conclusions

A cursory inspection of equation (19) reveals a very strong dependence of $\alpha$ on the radial distance, due to the assumptions of subsonic turbulence and hydrostatic equilibrium. The $\alpha$-parameterization of the viscosity hardly yields credible results. In a disk of $x_d = 100$, $\alpha$ varies by a factor of $3 \times 10^2$, compared with its value at $x = 1$. Assuming equality between the length scale of the turbulence and the height scale of the disk, because the disk is assumed to be thin, for $y = 0.01$, we have subsonic turbulence for only $x \leq 3.04$; for $y = 0.1$, for only $x \leq 23.56$. The results are highly dependent on the extent and thinness of the disk. The thicker the disk, the more subsonic it will be. It should be stressed that the conclusions we have drawn are based on the analysis of a solution obtained under conditions that, under the usual procedure, would result in the $\alpha$-standard model, which gives a disk scale height varying with $r^{9/8}$—a necessary condition for constant $\alpha$. However, the constancy of $\ell$ along $r$ is not an assumption but stems from a rigorous solution to the mass continuity equation. In other words, the assumption of constant $\alpha$ is not compatible with the solution to the mass continuity equation. To make things compatible, we should have

$$\frac{1}{r} \frac{\partial}{\partial r} (r \rho V_t) + L(\rho, V) = 0,$$  \hspace{1cm} (27)

where $L = L (\partial/\partial r, \partial/\partial z)$ is an operator applied to $\rho$ and $V$. If we insist in hydrostatic equilibrium, then $L$ describes the turbulent mass transport. It is beyond the scope of our Letter to go any further on this matter, but, if a disk height scale dependence on $r$ is essential to having physically meaningful results, an urgent search is indeed required for physical processes that contribute an extra term to the mass transport equation. In that respect, it is very unlikely that we would discard turbulent mass transport in the disk as one of the reasons for having the disk scale height vary along the radial distance, but, if we did, then turbulence would be the presumable source of viscosity in the disk. The points that we have raised in this
Letter deserve more attention, and in no way do they cover all aspects of this subject matter. It is important that we know the value of the disk height scale and that we understand the way that it behaves along $r$, in order to understand the relevant energy transport mechanism in a given region and how it varies along the disk. It is our intention to consider these issues, in a more detailed way, in a future article.

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